Universidade de Aveiro Departamento de Matemática
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2-restrictedly-regular hypermaps of small genus

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Tese apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, realizada sob a orientação científica do Dr. António João de Castilho Breda d'Azevedo, Professor Associado do Departamento de Matemática da Universidade de Aveiro

o júri

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Doutor António João de Castilho Breda d’Azevedo
Professor Associado da Universidade de Aveiro

Doutor Gareth Jones
Professor of the School of Mathematics of the University of Southampton

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palavras-chave

Hipermapa, restritamente-regular, 2-restritamente-regular, orientavelmente-regular, pseudo-orientavelmente-regular, bipartido-regular, grupo de quiralidade, índice de quiralidade.

resumo

Nesta tese consideramos hipermapas com grande número de automorfismos em superfícies de baixo género, nomeadamente a esfera, o plano projectivo, o toro e o duplo toro.

É conhecido o facto de que o número de automorfismos ou simetrias de um hipermapa H é limitado pelo seu número de flags, que, genericamente falando, são triplos vértice-aresta-face mutualmente incidentes. De facto, o número de automorfismos de H divide o número de flags de H. Hipermapas para os quais este limite é atingido são chamados regulares e estão classificados nas superfícies orientáveis até género 101 e em superfícies não-orientáveis até genero 202, usando computadores.

Neste trabalho classificamos os hipermapas 2-restritamente-regulares na esfera, no plano projectivo, no toro e no duplo toro, isto é, hipermapas cujo número de automorfismos é igual a metade do número de flags, e calculamos os seus grupos quiralidade e índices de quiralidade, que podem ser vistos como medidas algébricas e numéricas de quanto H se distancia de ser regular. Estes hipermapas são uma generalização dos hipermapas quirais.

Também introduzimos alguns métodos para construir hipermapas bipartidos. Duas destas construções têm um papel muito importante no nosso trabalho.

keywords

Hypermap, restrictedly-regular, 2-restrictedly-regular, orientably-regular, pseudo-orientably-regular, bipartite-regular, chirality group, chirality index.

abstract

This thesis deals with hypermaps having large automorphism group on surfaces of small genus, namely the sphere, the projective plane, the torus and the double torus.

It is well-known that the number of automorphisms or symmetries of a hypermap H is bounded by its number of flags, which are, roughly speaking, incident triples vertex-edge-face. In fact, the number of automorphisms of H divides the number of flags of H. Hypermaps for which this upper bound is attained are called regular and have been classified on orientable surfaces up to genus 101 and on non-orientable surfaces up to genus 202, using computers.

In this work we classify the 2-restrictedly-regular hypermaps on the sphere, the projective plane, the torus and the double torus, that is, hypermaps whose number of automorphism is equal to half the number of flags and compute their chirality groups and chirality indices, which may be regarded as algebraic and numerical measures of how far H deviates from being regular. These hypermaps are a generalization of chiral hypermaps.

We also introduce some methods for constructing bipartite hypermaps. Two of those constructions will play an important role in our work.

Contents

Introduction

This thesis deals with hypermaps having large automorphism group on surfaces of small genus, namely the sphere, the projective plane, the torus and the double torus.

Topologically, a hypermap $\mathcal{H}$ is a cellular imbedding of a connected hypergraph $\mathcal{G}$ into a compact surface $\mathcal{S}$. When $\mathcal{G}$ is a graph, we say that $\mathcal{H}$ is a map. The Euler characteristic and the genus of $\mathcal{H}$ are the Euler characteristic and the genus of $\mathcal{S}$. Roughly speaking, the flags of $\mathcal{H}$ are its incident triples vertex-edge-face, and a symmetry or an automorphism of $\mathcal{H}$ is a permutation of the set $\Omega_{\mathcal{H}}$ of flags of $\mathcal{H}$ preserving incidence. The set of all automorphisms of a hypermap $\mathcal{H}$ forms a permutation group, $\mathrm{Aut}(\mathcal{H})$, acting on the set of flags of $\mathcal{H}$. It has been shown [24] that every finite group is the group of automorphisms of a map (and hence of a hypermap). The number of automorphisms of a hypermap $\mathcal{H}$ is bounded by the number of flags of $\mathcal{H}$, since every automorphism is uniquely determined by its effect on a flag. In addition, the number of automorphisms of $\mathcal{H}$ divides the number of flags of $\mathcal{H}$. Hypermaps for which this upper bound is attained are called regular. Regular hypermaps may be thought of as a generalization of the Platonic solids. When $\mathcal{S}$ is orientable, $\mathcal{H}$ is said orientable and the number of automorphisms of $\mathcal{H}$ which induce orientation-preserving automorphisms of $\mathcal{S}$ is at most half the number of flags of $\mathcal{H}$. When the equality holds, the hypermap $\mathcal{H}$ is said orientably-regular. If $\mathcal{H}$ is orientably-regular but not regular, then $\mathcal{H}$ is chiral.

Algebraically, a hypermap $\mathcal{H}$ is completely determined by a hypermap subgroup H, which is a subgroup of the free product $\Delta = C_2 * C_2 * C_2$. The number of flags of $\mathcal{H}$ is equal to the index of H in Δ, and its automorphism group is isomorphic to $\mathrm{N}_\Delta(H)/H$, where $\mathrm{N}_\Delta(H)$ denotes the normalizer in Δ of H. The hypermap $\mathcal{H}$ is regular if H is normal in Δ, and is orientably-regular if H is normal in Δ^+, one of the seven normal subgroups of Δ of index 2. Following [8], we say that a hypermap is 2-restrictedly-regular if the normalizer $\mathrm{N}_\Delta(H)$ in Δ of a hypermap subgroup H is one of those seven subgroups of Δ. In other words, a hypermap is 2-restrictedly-regular if and only if its group of automorphisms acts on the set of flags with 2 orbits. These hypermaps can be view as a generalization of chiral or irreflexible hypermaps. For further reading on maps and hypermaps see [49, 45, 33, 28, 41, 46, 48, 13, 65], see also [23, 25, 26, 27] for the orientable case, and [16, 39] for maps and hypermaps with boundary.

The classification of all maps or hypermaps which satisfy a certain condition is a common problem in map and hypermap theory. Regular, orientably-regular and chiral maps and hypermaps have been classified according to genus or Euler characteristic [11, 12], number of edges or faces [70, 7, 10, 77, 54, 69, 40], or automorphism group [14]. Edge-transitive maps, that is, maps whose automorphism group acts transitively on the set of edges have been classified on the sphere (Grünbaum and Shephard [37]) and on the torus (Širáň, Tucker and Watkins [66]). Another problem is the determination of all g for which there is a map or hypermap of genus g with a certain property [21, 78].

Throughout the last century, many authors (Brahana [3], Threlfall [62], Sherk [55], Coxeter and Moser [33], Garbe [35], Bergau and Garbe [2]) worked on the classification of regular and orientably-regular maps without the help of computers. They all contributed to the classification of regular maps on orientable surfaces up to genus 7 and on non-orientable surfaces up to genus 8. The generalization to hypermaps was done by Corn and Singerman [28], Breda and Jones [15] and Breda [7] on orientable surfaces up to genus 2 and on non-orientable surfaces up to genus 4. It is well-known that the classification of regular maps and hypermaps on a non-orientable surface of genus g can be derived from the classification of regular maps and hypermaps on the orientable surface of genus $g-1$. Chiral maps were studied by Sherk [56], Garbe [35] and Wilson [75]. Breda and Nedela [11] classified all chiral hypermaps on surfaces up to genus 4. An almost complete classification of regular and chiral maps up to 100 edges can be found in [70, 69]. In [19], Conder and Dobcsányi give complete lists of all regular and chiral maps on orientable surfaces of genus 2 to 15, and all regular maps on non-orientable surfaces of genus 4 to 30 (that is, all regular and chiral maps on surfaces with Euler characteristic between -28 and -2). More recently, Conder [17] obtained lists of regular and chiral maps and hypermaps on orientable surfaces of genus 2 to 101 and regular maps and hypermaps on non-orientable surfaces of genus 2 to 202, up to isomorphism and duality, with the help of the new "LowIndexNormalSubgroups" routine in MAGMA [1].

In this thesis we determine, up to duality, all (isomorphism classes of) 2-restrictedly-regular hypermaps on the sphere, the projective plane, the torus and the double torus, and compute their chirality groups and chirality indices (see [6]).

In Chapter 1 we introduce the basic notation used throughout the text. We present methods for construction bipartite maps. Two of these constructions, Walsh and Pin, will play an important role in our thesis. The first is induced by Walsh's correspondence [67] between hypermaps and bipartite maps on the same surface. We also study the properties of the orientable double cover of a non-orientable hypermap $\mathcal{H}$, which is the smallest orientable hypermap covering $\mathcal{H}$ (see [13]).

Chapter 2 deals with 2-restrictedly-regular hypermaps on the sphere. Using the Euler formula, we see that there is an infinite number of possibilities for the valencies of the vertices, edges and faces of a regular or 2-restrictedly-regular hypermap on the sphere. In each case, there is exactly one regular or 2-restrictedly-regular hypermap with those valencies. We show that all 2-restrictedly-regular hypermaps on the sphere are obtained from regular hypermaps on the sphere using the Walsh or Pin constructions. Most of the content of this chapter is published in [9].

Chapter 3 deals with hypermaps on the projective plane. We determine the 2-restrictedly-regular hypermaps on the projective plane by inspecting the regular and 2-restrictedly-regular hypermaps on the sphere. As on the sphere, all 2-restrictedly-regular hypermaps on the projective plane are obtained from regular hypermaps on the projective plane using the Walsh or Pin constructions. There is an infinite number of possibilities for the valencies of the vertices, edges and faces of a regular or 2-restrictedly-regular hypermap on the projective plane. In each case, there is at most one regular or 2-restrictedly-regular hypermap with those valencies.

Hypermaps on the torus are studied in Chapter 4. Our main references are the work of Singerman and Syddall [57, 58] on uniform maps, and the work of Coxeter and Moser [33] on orientably-regular maps. On the torus, the Euler formula gives a finite number of possibilities for the valencies of the vertices, edges and faces of a regular or 2-restrictedly-regular hypermap, and in each case there is an infinite number of non-isomorphic regular and 2-restrictedly-regular hypermaps with those valencies. It is shown that the 2-restrictedly-regular hypermaps on the torus are either uniform or obtained from regular hypermaps on the torus using the Walsh and Pin constructions. We also introduce a notation for the uniform hypermaps on the torus.

Finally, in Chapter 5, we classify all 2-restrictedly-regular hypermaps on the double torus. Our work in this Chapter was influenced by [15].

At the end, we provide a subject index.

Chapter 1

Hypermaps

In this chapter we introduce basic terminology from the theory of hypermaps and at the same time establish our notation.

1.1 Definitions and notations

A *hypermap* is a four-tuple $\mathcal{H} = (\Omega_{\mathcal{H}}, h_0, h_1, h_2)$ where h_0, h_1, h_2 are permutations of a non-empty set $\Omega_{\mathcal{H}}$ such that ${h_0}^2 = {h_1}^2 = {h_2}^2 = 1$ and $\langle h_0, h_1, h_2 \rangle$ is transitive on $\Omega_{\mathcal{H}}$. The elements of $\Omega_{\mathcal{H}}$ are called *flags* of $\mathcal{H}$, the permutations h_0, h_1 and h_2 are called *canonical generators* of $\mathcal{H}$ and the group $\mathrm{Mon}(\mathcal{H}) = \langle h_0, h_1, h_2 \rangle$ is the *monodromy group* of $\mathcal{H}$. One says that $\mathcal{H}$ is a *map* if $(h_0 h_2)^2 = 1$. A hypermap is said *finite* if its set of flags is finite. If the permutations h_0, h_1 and h_2 are fixed-point free, we say that $\mathcal{H}$ *has no boundary* or that $\mathcal{H}$ is a hypermap *without boundary*. Henceforth, all hypermaps are to be finite and without boundary unless otherwise specified.

The *hypervertices* or 0-*faces* of $\mathcal{H}$ correspond to $\langle h_1, h_2 \rangle$-orbits on $\Omega_{\mathcal{H}}$. Likewise, the *hyperedges* or 1-*faces* and *hyperfaces* or 2-*faces* correspond to $\langle h_0, h_2 \rangle$- and $\langle h_0, h_1 \rangle$-orbits on $\Omega_{\mathcal{H}}$, respectively. If a flag ω belongs to the orbit determining a k-face f we say that ω belongs to f, or that f contains ω. We use the terms *vertices*, *edges* and *faces* instead of hypervertices, hyperedges and hyperfaces, for short. We denote the numbers of vertices, edges and faces of $\mathcal{H}$ by $V(\mathcal{H})$, $E(\mathcal{H})$ and $F(\mathcal{H})$. When just one hypermap, say $\mathcal{H}$, is under discussion, we omit the letter $\mathcal{H}$ from hypermap-theoretic symbols and write, for instance Ω, V, E and F instead of $\Omega_{\mathcal{H}}$, $V(\mathcal{H})$, $E(\mathcal{H})$ and $F(\mathcal{H})$.

Let $\{i, j, k\} = \{0, 1, 2\}$. We say that the k-face $f = \omega\langle R_i, R_j \rangle$ and the j-face $e = \sigma\langle R_i, R_k \rangle$ are *incident* if $f \cap e \neq \varnothing$. In other words, incidence is given by non-empty intersection. Two k-faces f and f' are *adjacent* if both are incident to a j-face g. The *valency* of a k-face $f = w\langle h_i, h_j \rangle$ (of a finite hypermap without boundary), where $\omega \in \Omega_{\mathcal{H}}$, is the least positive integer n such that $(h_i h_j)^n \in \mathrm{Stab}(w)$. Since ${h_i}^2 = {h_j}^2 = 1$ and h_i and h_j are fixed-point free, f has $2n$ elements, so the valency of a k-face is equal to half of its cardinality. If, for each choice of indices $i, j \in \{0, 1, 2\}$, all $\langle h_i, h_j \rangle$-orbits on $\Omega_{\mathcal{H}}$ have the same cardinality, we say that $\mathcal{H}$ is *uniform*. When all vertices, edges and faces of $\mathcal{H}$ have valency greater than one, we can think of a flag as an incident vertex-edge-face triple (v, e, f). A hypermap $\mathcal{H}$ has *type* (l, m, n) if l, m and n are the least common multiples of the valencies of the vertices, edges and faces, respectively. In other words, the type of a hypermap $\mathcal{H}$ is (l_0, l_1, l_2) if l_i, l_j and l_k are the orders of $h_j h_k$, $h_k h_i$ and $h_i h_j$. When $\mathcal{H}$ is uniform, $\mathcal{H}$ has type (l, m, n) if and only if

l, m and n are the valencies of the vertices, edges and faces of $\mathcal{H}$, respectively.

Topologically, maps and hypermaps can be represented by cellular imbeddings of connected graphs and hypergraphs into compact surfaces. A map $\mathcal{M}$ can be represented by a cellular imbedding of a connected graph $\mathcal{G}$ into a compact surface $\mathcal{S}$, where the vertices, edges and faces of the imbedding correspond to the vertices, edges and faces of $\mathcal{M}$. Using the well-known correspondence of Walsh between hypermaps and bipartite maps described in [67], we can represent a hypermap by a cellular imbedding of a bipartite graph (that is, a hypergraph) $\mathcal{G}$ into a compact surface $\mathcal{S}$, where the vertices of $\mathcal{G}$ correspond to the vertices and edges of $\mathcal{H}$ and two vertices of $\mathcal{G}$ are connected by an edge if and only if they form an incident pair vertex-edge of $\mathcal{H}$.

Alternatively, a hypermap $\mathcal{H}$ can be represented by a cellular imbedding of a connected trivalent graph $\mathcal{G}$ into a compact surface $\mathcal{S}$, together with a labelling of the faces with labels 0, 1 and 2 so that each edge of $\mathcal{G}$ is incident with two faces carrying different labels. In other words, $\mathcal{H}$ can be represented by the Schreier (right) coset graph (see §3.7 of [33], §7. of [64] or §4-3. of [68]) for the stabilizer of a flag $\omega \in \Omega_{\mathcal{H}}$ in the monodromy group of $\mathcal{H}$, $\mathrm{Mon}(\mathcal{H})$, with respect to the generators h_0, h_1 and h_2, with free edges replacing loops. The vertices of the graph $\mathcal{G}$ correspond to the flags of $\mathcal{H}$ and the faces labelled with k correspond to the k-faces of $\mathcal{H}$.

When $\mathcal{H}$ is represented by a cellular imbedding of a connected hypergraph $\mathcal{G}$ on a surface $\mathcal{S}$, we say that $\mathcal{G}$ is the *underlying hypergraph* of $\mathcal{H}$ and that $\mathcal{S}$ is the *underlying surface* of $\mathcal{H}$. A hypermap $\mathcal{H}$ has no boundary when its underlying surface $\mathcal{S}$ has no boundary. The *Euler characteristic* and the *genus* of a hypermap $\mathcal{H}$ are the Euler characteristic and the genus of its underlying surface $\mathcal{S}$, respectively. We speak of *characteristic* of $\mathcal{H}$, meaning the Euler characteristic of $\mathcal{H}$, for short. Hypermaps imbedded on the sphere are called *spherical*; hypermaps imbedded on the torus are called *toroidal*.

A *covering* from a hypermap $\mathcal{H} = (\Omega_{\mathcal{H}}, h_0, h_1, h_2)$ to another hypermap $\mathcal{G} = (\Omega_{\mathcal{G}}, g_0, g_1, g_2)$ is a function $\psi : \Omega_{\mathcal{H}} \to \Omega_{\mathcal{G}}$ that commutes according to the following diagram:

$$\begin{array}{ccc} \Omega_{\mathcal{H}} & \xrightarrow{h_i} & \Omega_{\mathcal{H}} \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \Omega_{\mathcal{G}} & \xrightarrow{g_i} & \Omega_{\mathcal{G}} \end{array},$$

that is, such that $h_i\psi = \psi g_i$ for all $i \in \{0, 1, 2\}$. Since $\mathrm{Mon}(\mathcal{G})$ acts transitively on $\Omega_{\mathcal{G}}$, ψ is surjective. Because $\mathrm{Mon}(\mathcal{H})$ acts transitively on $\Omega_{\mathcal{H}}$, the covering ψ is completely determined by the image of a flag of $\mathcal{H}$. By von Dyck's theorem ([42], p. 28) the assignment $h_i \mapsto g_i$ extends to a group epimorphism $\Psi : \mathrm{Mon}(\mathcal{H}) \to \mathrm{Mon}(\mathcal{G})$ called the *canonical epimorphism*. The covering ψ is an *isomorphism* if it is injective. If there is a covering ψ from $\mathcal{H}$ to $\mathcal{G}$, we say that $\mathcal{H}$ *covers* $\mathcal{G}$ or that $\mathcal{G}$ *is covered by* $\mathcal{H}$, and write $\mathcal{H} \to \mathcal{G}$; if ψ is an isomorphism we say that $\mathcal{H}$ *is isomorphic to* $\mathcal{G}$, or that $\mathcal{H}$ and $\mathcal{G}$ are *isomorphic*, and write $\mathcal{H} \cong \mathcal{G}$. When ψ is a covering from $\mathcal{H}$ to $\mathcal{G}$ and $|\Omega_{\mathcal{H}}| = 2|\Omega_{\mathcal{G}}|$ we say that ψ is a *double covering*. An *automorphism* or a *symmetry* of $\mathcal{H}$ is an isomorphism $\psi : \Omega_{\mathcal{H}} \to \Omega_{\mathcal{H}}$ from $\mathcal{H}$ to itself, that is, a function ψ that commutes with the canonical generators. Naturally, the set of all automorphisms (or symmetries) of $\mathcal{H}$ forms a group under composition, called the *automorphism group* of $\mathcal{H}$ and denoted by $\mathrm{Aut}(\mathcal{H})$. Since for all $\omega \in \Omega$, $(\omega\langle h_i, h_j\rangle)\psi = \omega\psi\langle g_i, g_j\rangle$, a covering $\psi : \Omega_{\mathcal{H}} \to \Omega_{\mathcal{G}}$ induces a surjective mapping between the set of k-faces of $\mathcal{H}$ and the set of k-faces of $\mathcal{G}$; an isomorphism induces a bijective correspondence between the set of k-faces of $\mathcal{H}$ and the set of

k-faces of $\mathcal{G}$. An automorphism ψ is called a *reflection* if there is a flag $\omega \in \Omega$ and $k \in \{0, 1, 2\}$ such that $\omega\psi = \omega r_k$.

Using the Euclidean Division Algorithm, one can easily show the following result.

Lemma 1.1.1. *Let $\psi : \Omega_{\mathcal{H}} \to \Omega_{\mathcal{G}}$ be a covering from $\mathcal{H}$ to $\mathcal{G}$ and $\omega \in \Omega_{\mathcal{H}}$. Then the valency of the k-face of $\mathcal{G}$ containing $\omega\psi$ divides the valency of the k-face of $\mathcal{H}$ containing ω.*

1.2 The triangle group

The free product

$$\Delta = C_2 * C_2 * C_2 = \langle R_0, R_1, R_2 \mid {R_0}^2 = {R_1}^2 = {R_2}^2 = 1 \rangle$$

is called the *triangle group*. By the torsion theorem for free products (Theorem 1.6 in §IV.1 of [51]), the conjugates of R_0, R_1 and R_2 are the only non-identity elements of finite order in Δ. More generally, for each triple $(l, m, n) \in (\mathbb{N} \cup \{\infty\})^3$, the *extended triangle group* is the group

$$\Delta(l, m, n) = \langle R_0, R_1, R_2 \mid {R_0}^2 = {R_1}^2 = {R_2}^2 = (R_1R_2)^l = (R_2R_0)^m = (R_0R_1)^n = 1 \rangle$$

where we regard equations of the form $(R_iR_j)^\infty = 1$ as being vacuous.

For positive integers l, m, n, the extended triangle group $\Delta(l, m, n)$ is the group generated by reflections in the sides of a triangle with angles π/l, π/m and π/n. This triangle will lie on the sphere, the Euclidean plane or the hyperbolic plane depending on whether $1/l+1/m+1/n$ is greater than, equal to or less than 1, respectively. It is well-known that:

- $\Delta(1, m, n) = \Delta(1, k, k) \cong D_k$, where $k = \gcd(m, n)$;
- $\Delta(2, 2, n) \cong D_n \times C_2$;
- $\Delta(2, 3, 3) \cong S_4$;
- $\Delta(2, 3, 4) \cong S_4 \times C_2$;
- $\Delta(2, 3, 5) \cong A_5 \times C_2$.

If N is a normal subgroup of Δ of index 2, then Δ/N, having order 2, is isomorphic to C_2. Consequently, the group Δ has 7 subgroups of index 2 (see [13]), the kernels of the $2^3 - 1 = 7$ group epimorphisms $\varphi : \Delta \to C_2$:

$$\Delta^+ = \langle R_1R_2, R_2R_0 \rangle^\Delta = \langle R_1R_2, R_2R_0, R_0R_1 \rangle,$$

$$\Delta^{\hat{k}} = \langle R_i, R_j \rangle^\Delta = \langle R_i, R_j, {R_i}^{R_k}, {R_j}^{R_k} \rangle,$$
$$\Delta^k = \langle R_k, R_iR_j \rangle^\Delta = \langle R_k, R_iR_j, R_jR_kR_i \rangle,$$

where $\{i, j, k\} = \{0, 1, 2\}$. The subgroup Δ^+ is often called the *even subgroup* of Δ.

If N is normal subgroup of Δ of index 4, then Δ/N, being a group of order 4 generated by reflections, is $V_4 \cong C_2 \times C_2$. By taking $\varphi : \Delta \to C_2 \times C_2$ a group epimorphism such that $N = \ker \varphi$, and π_1 and π_2 the projections $C_2 \times C_2 \to C_2$, one can see that $N_1 = \ker \varphi\pi_1$ and $N_2 = \ker \varphi\pi_2$ are normal subgroups of Δ of index 2 and $N = N_1 \cap N_2$. Consequently, the normal subgroups of Δ of index 4 are intersections of normal subgroups of Δ of index 2. By inspection we can see that Δ has 7 normal subgroups of index 4 (see [13]):

$$\begin{aligned}\Delta^{012} &= \langle R_iR_jR_k\rangle^{\Delta} = \langle R_iR_jR_k, R_jR_kR_i, R_kR_iR_j\rangle \\ &= \Delta^i \cap \Delta^j = \Delta^0 \cap \Delta^1 \cap \Delta^2,\end{aligned}$$

$$\begin{aligned}\Delta^{+k\hat{k}} &= \langle R_iR_j, (R_jR_k)^2\rangle^{\Delta} = \langle R_iR_j, (R_iR_j)^{R_k}, (R_jR_k)^2, (R_kR_i)^2\rangle \\ &= \Delta^+ \cap \Delta^k = \Delta^k \cap \Delta^{\hat{k}} = \Delta^{\hat{k}} \cap \Delta^+ = \Delta^+ \cap \Delta^k \cap \Delta^{\hat{k}},\end{aligned}$$

$$\begin{aligned}\Delta^{\hat{i}\hat{j}k} &= \langle R_k, (R_iR_j)^2\rangle^{\Delta} = \langle R_k, {R_k}^{R_i}, {R_k}^{R_j}, {R_k}^{R_iR_j}, (R_iR_j)^2\rangle \\ &= \Delta^{\hat{i}} \cap \Delta^{\hat{j}} = \Delta^{\hat{j}} \cap \Delta^k = \Delta^k \cap \Delta^{\hat{i}} = \Delta^{\hat{i}} \cap \Delta^{\hat{j}} \cap \Delta^k\end{aligned}$$

where $\{i,j,k\} = \{0,1,2\}$. We write $\Delta^{\hat{0}1\hat{2}}$ and $\Delta^{0\hat{1}\hat{2}}$ instead of $\Delta^{\hat{0}\hat{2}1}$ and $\Delta^{\hat{1}\hat{2}0}$, for simplicity.

Let Δ' be the derived group (that is, the commutator subgroup) of Δ. For all $i,j \in \{0,1,2\}$, $(R_iR_j)^2 = [R_i, R_j] \in \Delta'$, so the first homology group of Δ is $\Delta/\Delta' \cong C_2 \times C_2 \times C_2$ and $\Delta' = \langle (R_1R_2)^2, (R_2R_0)^2, (R_0R_1)^2\rangle^{\Delta} = \Delta^{\hat{0}} \cap \Delta^{\hat{1}} \cap \Delta^{\hat{2}}$ is a normal subgroup of Δ of index 8.

1.3 Hypermap subgroups

Given a group G, we denote by $\mathrm{Z}(G)$ the center of G. If H is a subgroup of G, then we denote by $\mathrm{N}_G(H)$, H^G and H_G, the normalizer, the normal closure and the core of H in G, respectively.

Each hypermap $\mathcal{H}$ gives rise to a transitive permutation representation $\rho_{\mathcal{H}} : \Delta \to \mathrm{Mon}(\mathcal{H})$, $R_i \mapsto h_i$ of the free product $\Delta = C_2 * C_2 * C_2$. The group Δ acts naturally and transitively on $\Omega_{\mathcal{H}}$ via $\rho_{\mathcal{H}}$. The stabilizer $H = \mathrm{Stab}_{\Delta}(\omega)$ of a flag $\omega \in \Omega_{\mathcal{H}}$ under the action of Δ is called the *hypermap subgroup* or *fundamental group* of $\mathcal{H}$. Since Δ acts transitively on $\Omega_{\mathcal{H}}$, hypermap subgroups are unique up to conjugation in Δ. The valency of a k-face containing ω is the least positive integer n such that $(R_iR_j)^n \in \mathrm{Stab}_{\Delta}(\omega) = H$; more generally, the valency of a k-face containing the flag $\sigma = \omega \cdot g = \omega(g)\rho_{\mathcal{H}} \in \Omega_{\mathcal{H}}$, where $g \in \Delta$, is the least positive integer n such that $(R_iR_j)^n \in \mathrm{Stab}_{\Delta}(\sigma) = \mathrm{Stab}_{\Delta}(\omega \cdot g) = \mathrm{Stab}_{\Delta}(\omega)^g = H^g$. We remark that a hypermap of type (l,m,n) can be regarded as a transitive permutation representation of the extended triangle group $\Delta(l,m,n)$ (see [13]).

Lemma 1.3.1. *Let $\mathcal{H}$ and $\mathcal{G}$ be hypermaps with hypermap subgroups H and G respectively. Then $\mathcal{H} \to \mathcal{G}$ if and only if $H \subseteq G^g$ for some $g \in \Delta$.*

Proof. Let $\omega \in \Omega_{\mathcal{H}}$ and $\sigma \in \Omega_{\mathcal{G}}$ such that $H = \mathrm{Stab}_{\Delta}(\omega)$ and $G = \mathrm{Stab}_{\Delta}(\sigma)$.
($\Rightarrow$) Let $\varphi : \Omega_{\mathcal{H}} \to \Omega_{\mathcal{G}}$ be a covering and $g \in \Delta$ such that $\omega\psi = \sigma g$. Then, for all $h \in \Delta$,

$$h \in H \Leftrightarrow \omega h = \omega \Rightarrow \sigma g h = \omega\psi h = \omega h\psi = \omega\psi = \sigma g \Leftrightarrow h \in \mathrm{Stab}_{\Delta}(\sigma g) = \mathrm{Stab}_{\Delta}(\sigma)^g = G^g,$$

that is, $H \subseteq G^g$.
($\Leftarrow$) If $H \subseteq G^g$, then $\varphi : \Omega_{\mathcal{H}} \to \Omega_{\mathcal{G}}$, $\omega h\varphi = \sigma g h$ is well defined and is a covering $\mathcal{H} \to \mathcal{G}$. □

Corollary 1.3.2. *Let $\mathcal{H}$ and $\mathcal{G}$ be hypermaps with hypermap subgroups H and G respectively. Then $\mathcal{H} \cong \mathcal{G}$ if and only if $H = G^g$ for some $g \in \Delta$. In other words, $\mathcal{H}$ and $\mathcal{G}$ are isomorphic if and only if there is an inner automorphism θ of Δ such that $H\theta = G$.*

This last result shows that there is a natural correspondence between the isomorphism classes of hypermaps and the conjugation classes of subgroups of Δ.

Let H be a hypermap subgroup of $\mathcal{H}$. Denote by $\mathrm{Alg}(H) = (\Delta /_r H, \cdot H_\Delta R_0, \cdot H_\Delta R_1, \cdot H_\Delta R_2)$ where $\cdot H_\Delta R_i : \Delta /_r H \to \Delta /_r H$, $Hg \mapsto HgH_\Delta R_i = HgR_i$. We say that $\mathrm{Alg}(H)$ is an *algebraic presentation* of $\mathcal{H}$.

Lemma 1.3.3. *Let* $\mathrm{Alg}(H)$ *be as above. Then* $\mathcal{H}$ *is isomorphic to* $\mathrm{Alg}(H)$*. Furthermore, the groups* $\mathrm{Mon}(\mathcal{H})$ *and* Δ/H_Δ *are isomorphic.*

This Lemma shows that, up to isomorphism, every hypermap $\mathcal{H}$ is completely determined by a hypermap subgroup H. For simplicity, we do not differentiate $\mathcal{H}$ from its algebraic presentations, and so we see, for instance, $\Omega_\mathcal{H}$ as $\Delta /_r H$ and $\mathrm{Mon}(\mathcal{H})$ as Δ/H_Δ, for some hypermap subgroup H of $\mathcal{H}$.

Lemma 1.3.4. *Let* $\mathcal{H}$ *be a hypermap,* $\omega \in \Omega_\mathcal{H}$ *and* $H = \mathrm{Stab}_\Delta(\omega)$ *a hypermap subgroup of* $\mathcal{H}$*. Then* $\mathrm{Aut}(\mathcal{H}) \cong \mathrm{N}_\Delta(H)/H$*. Moreover,* $h \in \mathrm{N}_\Delta(H)$ *if and only if for every flag* $Hg \in \Delta /_r H$ *there is an automorphism of* $\mathcal{H}$ *which maps* Hg *to* Hhg*.*

Note that an automorphism ψ is a reflection if and only if there is $g \in \Delta$ and $k \in \{0, 1, 2\}$ such that $R_k \in H^g$.

Of the two groups $\mathrm{Mon}(\mathcal{H})$ and $\mathrm{Aut}(\mathcal{H})$, the first acts transitively on Ω (by definition) and the second, due to the commutativity of the automorphisms with the canonical generators, acts semi-regularly on $\Omega_\mathcal{H}$. These two actions give rise to the following inequalities:

$$|\,\mathrm{Mon}(\mathcal{H})| \geq |\Omega_\mathcal{H}| \geq |\,\mathrm{Aut}(\mathcal{H})|. \tag{1.1}$$

Indeed, if H is a hypermap subgroup of $\mathcal{H}$, then $|\,\mathrm{Mon}(\mathcal{H})| = [\Delta : H_\Delta]$, $|\Omega_\mathcal{H}| = [\Delta : H]$ and $|\,\mathrm{Aut}(\mathcal{H})| = [\mathrm{N}_\Delta(H) : H]$.

Lemma 1.3.5. *The following statements are equivalent:*

1. $|\,\mathrm{Mon}(\mathcal{H})| = |\Omega_\mathcal{H}|$*, that is,* $\mathrm{Mon}(\mathcal{H})$ *acts regularly on* $\Omega_\mathcal{H}$*;*

2. $|\Omega_\mathcal{H}| = |\,\mathrm{Aut}(\mathcal{H})|$*, that is,* $\mathrm{Aut}(\mathcal{H})$ *acts regularly on* $\Omega_\mathcal{H}$*;*

3. $\mathcal{H}$ *has a hypermap subgroup which is normal in* Δ*.*

If $\mathrm{Mon}(\mathcal{H})$ or $\mathrm{Aut}(\mathcal{H})$ act regularly on $\Omega_\mathcal{H}$, or equivalently, if $\mathcal{H}$ has a hypermap subgroup which is normal in Δ, then $\mathcal{H}$ is said *regular*. It is well-known that every regular hypermap is uniform but the converse is not true. In Chapter 4 we can find uniform hypermaps which are not regular.

Let H be a hypermap subgroup of a hypermap $\mathcal{H}$. Following [8], if $H \leq \Theta$ for some $\Theta \lhd \Delta$, we say that $\mathcal{H}$ is Θ-*conservative*. We say that $\mathcal{H}$ is

- *orientable* if $\mathcal{H}$ is Δ^+-conservative,
- *bipartite* if $\mathcal{H}$ is $\Delta^{\hat{0}}$-conservative,
- *pseudo-orientable* if $\mathcal{H}$ is Δ^0-conservative[1].

[1] This extends Wilson's definition of pseudo-orientability [71] from maps to hypermaps.

Moreover, given $k \in \{0,1,2\}$, we say that $\mathcal{H}$ is *k-bipartite* if $\mathcal{H}$ is $\Delta^{\hat{k}}$-conservative, and *k-pseudo-orientable* if $\mathcal{H}$ is Δ^k-conservative. In addition, a k-bipartite hypermap is also called *vertex-bipartite* if $k=0$, *edge-bipartite* if $k=1$, and *face-bipartite* if $k=2$.

A hypermap $\mathcal{H}$ is orientable if and only if its underlying surface is orientable. Since $\Delta^+ \cap \Delta^{\hat{i}} = \Delta^+ \cap \Delta^i = \Delta^{\hat{i}} \cap \Delta^{\hat{i}}$ (see Section 1.2), an orientable hypermap $\mathcal{H}$ is $\Delta^{\hat{k}}$-conservative if and only if $\mathcal{H}$ is Δ^k-conservative; a non-orientable hypermap cannot be simultaneously $\Delta^{\hat{k}}$-conservative and Δ^k-conservative. A hypermap $\mathcal{H}$ is bipartite if and only if we can divide its set of vertices into two parts so that consecutive vertices around an edge or a face are in alternate parts, that is, if for all $\omega \in \Omega_{\mathcal{H}}$, the vertices containing ω and ωh_0 are in different parts. A hypermap $\mathcal{H}$ is pseudo-orientable if we can give orientations to the vertices so that consecutive vertices around an edge or a face have different orientations, that is, if for all $\omega \in \Omega_{\mathcal{H}}$, the vertices containing ω and ωh_0 have different orientations.

Lemma 1.3.6. *If $\mathcal{H}$ is bipartite or pseudo-orientable, then all edges and all faces have even valencies.*

Proof. Let Θ be $\Delta^{\hat{0}}$ or Δ^0, $\mathcal{H}$ a Θ-conservative hypermap, $\omega \in \Omega_{\mathcal{H}}$, and $H = \mathrm{Stab}_\Delta(\omega)$. If m and n are the valencies of the edge and the face containing the flag ωg, then $(R_2R_0)^m, (R_0R_1)^n \in \mathrm{Stab}_\Delta(\omega g) = H^g \subseteq \Theta^g = \Theta$. In both cases m and n must be even. □

Let Θ be a normal subgroup of Δ and $\mathcal{H}$ a Θ-conservative hypermap. An automorphism $\varphi \in \mathrm{Aut}(\mathcal{H})$ is said *Θ-conservative* if it preserves the Θ-orbits on $\Omega_{\mathcal{H}} = \Delta/_r H$, that is, if for all $Hg \in \Delta/_r H$, Hg and $(Hg)\varphi$ are in the same Θ-orbit. Since Θ is a normal subgroup of Δ containing H, Θ contains H_Δ and so Θ/H_Δ is a normal subgroup of $\Delta/H_\Delta = \mathrm{Mon}(\mathcal{H})$. Since every covering is determined by the image of a flag, we get the following result.

Lemma 1.3.7. *Let Θ be a normal subgroup of Δ and $\mathcal{H}$ a Θ-conservative hypermap with hypermap subgroup H. An automorphism φ of $\mathcal{H}$ is Θ-conservative if and only if $H\varphi \in H \cdot \Theta/H_\Delta$.*

Proof. Only the necessary condition needs to be proved. Let $H\varphi = Ht$, with $t \in \Theta$. Then, for all $g \in \Delta$, $t^g \in \Theta^g = \Theta$ and $(Hg)\varphi = H\varphi g = Htg = Hgt^g \in Hg \cdot \Theta/H_\Delta$. □

The set of all Θ-conservative automorphisms of a Θ-conservative hypermap $\mathcal{H}$ forms a group under composition denoted by $\mathrm{Aut}^\Theta(\mathcal{H})$. The groups of Δ^+- and $\Delta^{+0\hat{0}}$-conservative automorphisms of $\mathcal{H}$ are also denoted by $\mathrm{Aut}^+(\mathcal{H})$ and $\mathrm{Aut}^{+0\hat{0}}(\mathcal{H})$, respectively.

Now let Θ be a normal subgroup of Δ of index 2. Then every Θ-conservative hypermap $\mathcal{H}$ has exactly two Θ-orbits. An automorphism φ of $\mathcal{H}$ is called *Θ-preserving* if φ stabilizes the two orbits, and is called *Θ-reversing* if φ interchanges the two orbits. We also say that an automorphism φ of an orientable hypermap is *orientation-preserving* if φ is Δ^+-reversing, and *orientation-reversing* if φ is Δ^+-reversing. The group of orientation-preserving automorphisms of an orientable hypermap $\mathcal{H}$, $\mathrm{Aut}^+(\mathcal{H})$, is often called the *rotation group* of $\mathcal{H}$.

When $H \lhd \Theta$, $\mathcal{H}$ is called *Θ-regular*. If $\mathcal{H}$ is Θ-regular but not regular, $\mathcal{H}$ is called *Θ-chiral*. We say that $\mathcal{H}$ is *orientably-regular* if $\mathcal{H}$ is Δ^+-regular, *orientably-chiral* if $\mathcal{H}$ is Δ^+-chiral, *bipartite-regular* if $\mathcal{H}$ is $\Delta^{\hat{0}}$-regular, *bipartite-chiral* if $\mathcal{H}$ is $\Delta^{\hat{0}}$-chiral, *pseudo-orientably-regular* if $\mathcal{H}$ is Δ^0-regular and *pseudo-orientably-chiral* if $\mathcal{H}$ is Δ^0-chiral.

More generally, given $k \in \{0,1,2\}$, we say that $\mathcal{H}$ is *k-bipartite-regular* if $\mathcal{H}$ is $\Delta^{\hat{k}}$-regular, *k-bipartite-chiral* if $\mathcal{H}$ is $\Delta^{\hat{k}}$-chiral, *k-pseudo-orientably-regular* if $\mathcal{H}$ is Δ^k-regular, and *k-pseudo-orientably-chiral* if $\mathcal{H}$ is Δ^k-chiral. A k-bipartite-regular (resp. k-bipartite-chiral) hypermap is also called *vertex-bipartite-regular* (resp. *vertex-bipartite-chiral*) if $k = 0$, *edge-bipartite-regular* (resp. *edge-bipartite-chiral*) if $k = 1$, and *face-bipartite-regular* (resp. *face-bipartite-chiral*) if $k = 2$.

The group of Θ-conservative automorphisms of a Θ-conservative hypermap $\mathcal{H}$, $\mathrm{Aut}^{\Theta}(\mathcal{H})$, is isomorphic to $\mathrm{N}_{\Theta}(H)/H$. When $\mathcal{H}$ is Θ-regular, $\mathrm{N}_{\Theta}(H) = \Theta$ and so $\mathrm{Aut}^{\Theta}(\mathcal{H})$ is isomorphic to Θ/H. The hypermap $\mathcal{H}$ is Θ-regular if and only if its Θ-conservative automorphism group $\mathrm{Aut}^{\Theta}(\mathcal{H})$ acts transitively on each Θ-orbit in $\Omega_{\mathcal{H}}$.

A hypermap $\mathcal{H}$ is *rotary* (see [72] for maps) if there is $\omega \in \Omega_{\mathcal{H}}$ and $\upsilon, \varphi \in \mathrm{Aut}(\mathcal{H})$ with the property that υ and φ cyclically permute the consecutive edges incident to the vertex v and the face f containing ω, respectively. In other words, a hypermap is rotary if the normalizer in Δ of a hypermap subgroup contains Δ^+. An orientable hypermap $\mathcal{H}$ is rotary if and only if $\mathcal{H}$ is orientably-regular; a non-orientable hypermap $\mathcal{H}$ is rotary if and only if $\mathcal{H}$ is regular (see [33, 72] for maps). A hypermap $\mathcal{H}$ is said *reflexible* if its automorphism group has an orientation-reversing automorphism and *chiral* or *irreflexible* otherwise ([33, 49]). Orientably-regular maps and hypermaps have often been called "regular" [3, 33, 28, 25, 26, 27], while regular maps and hypermaps have been called "reflexible" [33].

Following [8], a hypermap $\mathcal{H}$ is called *restrictedly-regular* if $\mathcal{H}$ is Θ-regular for some normal subgroup Θ with finite index in Δ. If $H \lhd \Theta$ and $\Theta \lhd \Delta$, then

$$H \subseteq \Theta \subseteq (\mathrm{N}_{\Delta}(H))_{\Delta} \subseteq \mathrm{N}_{\Delta}(H),$$

that is, when $\mathcal{H}$ is restrictedly-regular, the subgroup $(\mathrm{N}_{\Delta}(H))_{\Delta}$, called *regularity-subgroup* of $\mathcal{H}$, is the largest normal subgroup of Δ in which H is normal.

More generally, we say that $\mathcal{H}$ is *k-restrictedly-regular* if k is the index of the regularity-subgroup of $\mathcal{H}$ in Δ, that is, if $k = [\Delta : (\mathrm{N}_{\Delta}(H))_{\Delta}]$. The index k is called the *restricted rank* of $\mathcal{H}$. Since

$$\begin{aligned}
|\Omega_{\mathcal{H}}| = [\Delta : H] &= [\Delta : (\mathrm{N}_{\Delta}(H))_{\Delta}] \cdot [(\mathrm{N}_{\Delta}(H))_{\Delta} : H] \\
&= k \cdot [(\mathrm{N}_{\Delta}(H))_{\Delta} : H] \\
&\leq k \cdot [\mathrm{N}_{\Delta}(H) : (\mathrm{N}_{\Delta}(H))_{\Delta}] \cdot [(\mathrm{N}_{\Delta}(H))_{\Delta} : H] \\
&= k \cdot [\mathrm{N}_{\Delta}(H) : H] \\
&= k \cdot |\mathrm{Aut}(\mathcal{H})|,
\end{aligned}$$

when $\mathcal{H}$ is k-restrictedly-regular, $|\Omega_{\mathcal{H}}|/|\mathrm{Aut}(\mathcal{H})| \leq k$ and $k \mid |\Omega_{\mathcal{H}}|$. The restricted rank of a hypermap $\mathcal{H}$ can be regarded as a numerical measure of how far $\mathcal{H}$ deviates from being regular.

A 1-restrictedly-regular hypermap is a regular hypermap; a 2-restrictedly-regular hypermap is a Θ-chiral hypermap, where Θ is 1 of the 7 normal subgroups of Δ of index 2.

Lemma 1.3.8. *A hypermap is 2-restrictedly-regular if and only if the number of automorphisms of $\mathcal{H}$ is equal to half the number of flags.*

In [47], Jones called a map $\mathcal{M}$ *just-edge-transitive* if $\mathcal{M}$ is 4-restrictedly-regular and its regularity subgroup is $\Delta^{\hat{0}1\hat{2}}$. The classification of Δ^{012}-regular hypermaps of small genus, as well as their chirality groups and chirality indices can be found in [5].

The types automorphism groups of edge-transitive maps, which include all 2-restrictedly-regular maps except the $\Delta^{\hat{1}}$-chiral, were classified by Wilson in [76] and Graver and Watkins

an edge-transitive map with automorphism group of type ... (Wilson)	an edge-transitive map with automorphism group of type ... (Graver & Watkins)	regularity--subgroups
I	1	Δ
IIa	$2^{\mathsf{P}}\mathsf{ex}$	Δ^{+}
IIb	$2\mathsf{ex}$	Δ^{2}
	$2^{*}\mathsf{ex}$	Δ^{0}
IIc	2	$\Delta^{\hat{0}}$
	2^{*}	$\Delta^{\hat{2}}$
IId	2^{P}	Δ^{1}
IIIa	3	$\Delta^{\hat{0}1\hat{2}}$
IIId	5	$\Delta^{+0\hat{0}}$
	5^{*}	$\Delta^{+2\hat{2}}$
IIIe	5^{P}	Δ^{012}

Table 1.1: Correspondence between edge-transitive maps and restrictedly-regular maps.

in [36]. In Table 1.1 we give the correspondence between types of edge-transitive maps of Wilson and of Graver and Watkins, and their regularity-subgroups.

Let Θ be a normal subgroup of Δ. The hypermap with hypermap subgroup Θ is called the *trivial* Θ*-hypermap* and denoted by $\mathcal{T}_\Theta$. It is a regular hypermap with $[\Delta : \Theta]$ flags which may have boundary. In §5 of [13], Breda and Jones classify the 16 trivial Θ-hypermaps with abelian automorphism group. Their hypermap subgroups are the 16 normal subgroups of Δ containing Δ' (see Section 1.2). By Lemma 1.3.1, a hypermap $\mathcal{H}$ is Θ-conservative if and only if $\mathcal{H}$ covers $\mathcal{T}_\Theta$. Let $\mathcal{H}$ be a Θ-conservative hypermap, φ a covering from $\mathcal{H}$ to $\mathcal{T}_\Theta$ and $\{v_1, \dots, v_p\}$, $\{e_1, \dots, e_q\}$, $\{f_1, \dots f_r\}$ the sets of vertices, edges, faces of $\mathcal{T}_\Theta$, respectively. We recall that φ maps k-faces of $\mathcal{H}$ to k-faces of $\mathcal{T}_\Theta$. We say that $\mathcal{H}$ is Θ*-uniform* if for all $k \in \{0, 1, 2\}$, all k-faces of $\mathcal{H}$ mapped to a k-face of $\mathcal{T}_\Theta$ have the same valency. To put it another way, a Θ-conservative hypermap $\mathcal{H}$ is Θ-uniform if for all $k \in \{0, 1, 2\}$, k-faces containing flags in the same Θ-orbit have the same valency. When $\mathcal{H}$ is a Θ-uniform hypermap such that all vertices of $\mathcal{H}$ mapped to the vertex v_i of $\mathcal{T}_\Theta$ have valency l_i, all edges of $\mathcal{H}$ mapped to the edge e_j of $\mathcal{T}_\Theta$ have valency m_j and all faces of $\mathcal{H}$ mapped to the face f_k of $\mathcal{T}_\Theta$ have valency n_k, we say that $\mathcal{H}$ has Θ*-type* $(l_1, \dots, l_p; m_1, \dots m_q; n_1, \dots n_r)$. We may assume, without loss of generality, that $l_1 \leq \dots \leq l_p$, $m_1 \leq \dots \leq m_q$ and $n_1 \leq \dots \leq n_r$. A hypermap is called *bipartite-uniform* if it is $\Delta^{\hat{0}}$-uniform. The *bipartite-type* of a bipartite-uniform hypermap $\mathcal{B}$ is its $\Delta^{\hat{0}}$-type $(l_1, l_2; m; n)$, where l_1 and l_2 are the valencies (not necessarily distinct) of the vertices of $\mathcal{B}$, and m and n are the valencies of the edges and the faces of $\mathcal{B}$. Since $\mathcal{B}$ is bipartite-uniform, $\mathcal{B}$ is bipartite and, by Lemma 1.3.6, m and n are even. Moreover, a $\Delta^{\hat{k}}$-uniform hypermap is called k*-bipartite-uniform*; we also use the terms *vertex-bipartite-uniform*, *edge-bipartite-uniform* and *face-bipartite-uniform* instead of 0-bipartite-uniform, 1-bipartite-uniform and 2-bipartite-uniform, respectively.

Lemma 1.3.9. *Let Θ be a normal subgroup of Δ and $\mathcal{H}$ a Θ-conservative hypermap.*

1. *If $\mathcal{H}$ is Θ-regular, then $\mathcal{H}$ is Θ-uniform.*

2. *If Θ is Δ^+, Δ^0, Δ^1, Δ^2 or Δ^{012}, then $\mathcal{H}$ is Θ-uniform if and only if $\mathcal{H}$ is uniform.*

3. *If Θ is $\Delta^{+0\hat{0}}$, then $\mathcal{H}$ is $\Delta^{+0\hat{0}}$-uniform if and only if $\mathcal{H}$ is bipartite-uniform.*

Proof. 1. Let $k \in \{0,1,2\}$, $\omega \in \Omega_{\mathcal{H}}$, $H = \mathrm{Stab}_{\Delta}(\omega)$ and $g \in \Theta$. If $\mathcal{H}$ is Θ-regular, then $H \lhd \Theta$ and hence $H^g = H$. In particular, the k-faces containing ω and ωg have the same valency.
2. and 3. One can easily see that the hypermaps $\mathcal{T}_{\Theta}$, where Θ is Δ^+, Δ^0, Δ^1 or Δ^2, have 1 vertex, 1 edge and 1 face; the hypermaps $\mathcal{T}_{\Delta^{\hat{0}}}$ and $\mathcal{T}_{\Delta^{+0\hat{0}}}$ have 2 vertices, 1 edge and 1 face. □

A uniform hypermap is k-bipartite-uniform if and only if it is k-bipartite. Examples of Θ-uniform hypermaps that are not Θ-regular can be found in Chapter 4.

1.4 The Euler formula and the Hurwitz bound

A theorem of Hurwitz [38] (cf. [27, 18, 61]) states that an upper bound for the number of conformal automorphisms of a compact Riemann surface with genus g greater than one (that is, homeomorphisms of the surface onto itself preserving the local structure) is $84(g-1)$. It has been proved by Jones and Singerman [49] that the group of orientation-preserving automorphisms of a map $\mathcal{M}$ on an orientable surface of genus g is isomorphic to a group of conformal automorphisms of a compact Riemann surface with the same genus, and hence bounded by $84(g-1)$. Moreover, the number of automorphism of a map $\mathcal{M}$ is bounded by $168(g-1)$, if $\mathcal{M}$ is orientable, and by $84(g-2)$, otherwise (see, for instance, Theorem 4.2.2 of [61]).

Our aim in this section is to present methods for finding all possible types (resp. bipartite-types) of uniform (resp. bipartite-uniform) hypermaps on a given surface. We give a relation between the Euler characteristic, number of flags and type (resp. bipartite-type) of a uniform (resp. bipartite-uniform) hypermap, and then we use it to find bounds for the numbers of flags of uniform (resp. bipartite-uniform) hypermaps with a given negative Euler characteristic.

Using the well-known Euler (polyhedral) formula one can easily get the following result.

Lemma 1.4.1 (Euler formula for hypermaps)**.** *Let $\mathcal{H}$ be a hypermap with V vertices, E edges, F faces and Euler characteristic χ. Then*

$$\chi = V + E + F - \frac{|\Omega_{\mathcal{H}}|}{2}. \tag{1.2}$$

When $\mathcal{H}$ is uniform of type (l,m,n), $V = |\Omega_{\mathcal{H}}|/2l$, $E = |\Omega_{\mathcal{H}}|/2m$ and $F = |\Omega_{\mathcal{H}}|/2n$. Replacing the values of V, E and F in formula (1.2), we get:

Corollary 1.4.2 (Euler formula for uniform hypermaps)**.** *Let $\mathcal{H}$ be a uniform hypermap of type (l,m,n) with Euler characteristic χ. Then*

$$\chi = \frac{|\Omega_{\mathcal{H}}|}{2}\left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1\right). \tag{1.3}$$

When $\mathcal{H}$ is bipartite-uniform of bipartite-type $(l_1, l_2; m; n)$, each $\Delta^{\hat{0}}$-orbit has $|\Omega_{\mathcal{H}}|/2$ flags, and so the numbers of vertices in the $\Delta^{\hat{0}}$-orbits are $|\Omega_{\mathcal{H}}|/4l_1$ and $|\Omega_{\mathcal{H}}|/4l_2$. Then $\mathcal{H}$ has $V = |\Omega_{\mathcal{H}}|/4l_1 + |\Omega_{\mathcal{H}}|/4l_2$ vertices, $E = |\Omega_{\mathcal{H}}|/2m$ edges and $F = |\Omega_{\mathcal{H}}|/2n$ faces. Replacing the values of V, E and F in formula (1.2), we get:

Corollary 1.4.3 (Euler formula for bipartite-uniform hypermaps). *Let $\mathcal{H}$ be a bipartite-uniform hypermap of bipartite-type $(l_1, l_2; m; n)$ with Euler characteristic χ. Then*

$$\chi = \frac{|\Omega_{\mathcal{H}}|}{2}\left(\frac{1}{2l_1} + \frac{1}{2l_2} + \frac{1}{m} + \frac{1}{n} - 1\right). \tag{1.4}$$

Lemma 1.4.4. *If $\mathcal{H}$ is a hypermap such that all vertices have valency 1, then $\mathcal{H}$ is a uniform hypermap on the sphere of type $(1, k, k)$, where k is the number of vertices. Furthermore, $\mathcal{H}$ is regular.*

Proof. If all vertices have valency 1, then $R_1R_2 \in H^g$, for all $g \in \Delta$, so $R_1R_2 \in H_\Delta$. Consequently, $H_\Delta R_1 = H_\Delta R_2$ and

$$\mathrm{Mon}(\mathcal{H}) = \Delta/H_\Delta = \langle H_\Delta R_0, H_\Delta R_1, H_\Delta R_2\rangle = \langle H_\Delta R_2, H_\Delta R_0\rangle = \langle H_\Delta R_0, H_\Delta R_1\rangle. \tag{1.5}$$

Since $\mathrm{Mon}(\mathcal{H})$ acts transitively on $\Omega_{\mathcal{H}}$, $\mathcal{H}$ has exactly one $\langle H_\Delta R_2, H_\Delta R_0\rangle$-orbit and one $\langle H_\Delta R_0, H_\Delta R_1\rangle$-orbit, that is, 1 edge and 1 face, both with valencies $k := |\Omega_{\mathcal{H}}|/2$. Obviously, $\mathcal{H}$ is uniform of type $(1, k, k)$ and has k vertices, 1 edge and 1 face. Finally, using the Euler formula for hypermaps (Lemma 1.4.1), we see that $\chi_{\mathcal{H}} = V + E + F - |\Omega_{\mathcal{H}}|/2 = |\Omega_{\mathcal{H}}|/2 + 1 + 1 - |\Omega_{\mathcal{H}}|/2 = 2$. □

Now assume that $\mathcal{H}$ is a uniform hypermap of type (l, m, n). By Corollary 1.4.2, $\mathcal{H}$ is imbedded on a surface with Euler characteristic greater than, equal to, or smaller than 0 depending on whether $1/l + 1/m + 1/n$ is greater than, equal to, or smaller than 1, respectively.

Lemma 1.4.5. *Let l, m, n be positive integers such that $l \le m \le n$, and $S = \frac{1}{l} + \frac{1}{m} + \frac{1}{n}$. Then*

1. *$S > 1$ if and only if (l, m, n) is $(1, j, k)$, $(2, 2, k)$, $(2, 3, 3)$, $(2, 3, 4)$ or $(2, 3, 5)$, $j, k \in \mathbb{N}$;*
2. *$S = 1$ if and only if (l, m, n) is $(2, 3, 6)$, $(2, 4, 4)$ or $(3, 3, 3)$;*
3. *$S < 1$ if and only if $S \le \frac{1}{2} + \frac{1}{3} + \frac{1}{7} = \frac{41}{42}$.*

Proof. 1. When $S > 1$, $3/l \ge S > 1$, and so $l < 3$. If $l = 1$, then $S > 1$; else, if $l = 2$, then $2/m \ge 1/m + 1/n > 1/2$ and hence $m < 4$. Then $m = 2$, or $m = 3$ and $n < 6$.
2. When $S = 1$, $3/l \ge S = 1 > 1/l$, and so $1 < l \le 3$. If $l = 2$, then $2/m \ge 1/m + 1/n = 1/2 > 1/m$, so $2 < m \le 4$ and $n = 2m/(m-2)$. Then $m = 3$ and $n = 6$, or $m = n = 4$. If $l = 3$, then $1 = 3/l \ge S = 1$ implies that $l = m = n = 3$.
3. Assume that l, m, n are positive integers such that $l \le m \le n$ and $S < 1$. Then:
(a) if $l = 2$, $m = 3$ and $n > 6$, then $S \le \frac{1}{2} + \frac{1}{3} + \frac{1}{7} = \frac{41}{42}$;
(b) if $l = 2$, $m = 4$ and $n > 4$, then $S \le \frac{1}{2} + \frac{1}{4} + \frac{1}{5} = \frac{19}{20}$;
(c) if $l = 2$ and $m > 4$, then $S \le \frac{1}{2} + \frac{1}{5} + \frac{1}{5} = \frac{9}{10}$;
(d) if $l = 3$ and $n > 3$, then $S \le \frac{1}{3} + \frac{1}{3} + \frac{1}{4} = \frac{11}{12}$;
(e) if $l > 3$, then $S \le \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$. □

Using Corollary 1.4.2 together with Lemma 1.4.5, we get the following well-known result.

Theorem 1.4.6 (Hurwitz bound for uniform hypermaps with negative Euler characteristic). *If $\mathcal{H}$ is a uniform hypermap with negative Euler characteristic χ, then $|\Omega_{\mathcal{H}}| \le -84\chi$.*

Now we determine bounds for the number of flags of a bipartite-uniform hypermap with given negative Euler characteristic.

Let $\mathcal{B}$ be a bipartite-uniform hypermap of type $(l_1, l_2; m; n)$. According to Lemma 1.3.6, m and n are even. Let $(a, b, c, d) = (l_1, l_2, m/2, n/2)$. By Corollary 1.4.3, $\mathcal{H}$ is imbedded on a surface with Euler characteristic > 0, $= 0$ or < 0 depending on whether $1/a+1/b+1/c+1/d$ is greater than, equal to, or smaller than 2, respectively.

Lemma 1.4.7. *Let a, b, c and d be positive integers such that $a \leq b \leq c \leq d$, and $T = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$. Then:*

1. *$T > 2$ if and only if (a, b, c, d) is $(1, 1, j, k)$, $(1, 2, 2, k)$, $(1, 2, 3, 3)$, $(1, 2, 3, 4)$ or $(1, 2, 3, 5)$, where $j, k \in \mathbb{N}$;*
2. *$T = 2$ if and only if (a, b, c, d) is $(1, 2, 3, 6)$, $(1, 2, 4, 4)$, $(1, 3, 3, 3)$ or $(2, 2, 2, 2)$;*
3. *$T < 2$ if and only if $T \leq \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{7} = \frac{83}{42}$.*

Proof. Let $S = \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$. Then:
(a) if $a = 1$, then $T > 2$, $= 2$ or < 2 if and only if $S > 1$, $= 1$ or < 1, respectively;
(b) if $a = b = c = d = 2$, then $T = 2$;
(c) if $a = 2$ and $d > 2$, then $T \leq \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$;
(d) if $a > 2$, then $T \leq \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{4}{3}$.
Now the result follows from Lemma 1.4.5. □

Finally, using Corollary 1.4.3 together with Lemma 1.4.7, we get:

Theorem 1.4.8 (Hurwitz bound for bipartite-uniform hypermaps with negative Euler characteristic)**.** *If $\mathcal{H}$ is a bipartite-uniform hypermap with negative Euler characteristic χ, then $|\Omega_{\mathcal{H}}| \leq -168\chi$.*

1.5 Duality

Every automorphism θ of Δ gives rise to an operation on hypermaps by transforming a hypermap $\mathcal{H}$ with hypermap subgroup H, to its *operation-dual*, $\mathrm{D}_\theta(\mathcal{H})$, with hypermap subgroup $H\theta$ (see [41, 43, 44] for more details), that is, if $\mathcal{H} = (\Delta/_{\!r} H, H_\Delta R_0, H_\Delta R_1, H_\Delta R_2)$, then

$$\begin{aligned} \mathrm{D}_\theta(\mathcal{H}) &= (\Delta/_{\!r} H\theta, (H\theta)_\Delta R_0, (H\theta)_\Delta R_1, (H\theta)_\Delta R_2) \\ &= (\Delta/_{\!r} H\theta, H_\Delta \theta R_0, H_\Delta \theta R_1, H_\Delta \theta R_2). \end{aligned}$$

When θ is an inner automorphism, H and $H\theta$ are conjugate in Δ and, by Corollary 1.3.2, $\mathcal{H}$ and $\mathrm{D}_\theta(\mathcal{H})$ are isomorphic. Each permutation $\sigma \in S_{\{0,1,2\}}$ induces an outer automorphism (that is, a non-inner automorphism) $\overline{\sigma} : \Delta \to \Delta$ such that $R_i\overline{\sigma} = R_{i\sigma}$, for all $i = 0, 1, 2$. By abuse of language, we speak of D_σ, meaning the operator $\mathrm{D}_{\overline{\sigma}}$. These operations, presented by Machì in [52], transform one hypermap $\mathcal{H}$ to another by renaming its vertices, edges and faces. To be more precise, the k-face of $\mathcal{H}$ containing the flag Hg corresponds to the $k\sigma$-face of $\mathrm{D}_\sigma(\mathcal{H})$ containing $H\overline{\sigma}g\overline{\sigma}$. In particular, they have the same valency. James [41] showed that the operations on hypermaps form an infinite group, $\mathrm{Out}(\Delta)$, isomorphic to $PGL_2(\mathbb{Z})$ containing Machì's operations.

Lemma 1.5.1. *Let $\sigma \in S_{\{0,1,2\}}$ and $\overline{\sigma} : \Delta \to \Delta$ defined as above. Then $\Delta^+\overline{\sigma} = \Delta^+$, $\Delta^{\hat{k}}\overline{\sigma} = \Delta^{\widehat{k\sigma}}$ and $\Delta^k\overline{\sigma} = \Delta^{k\sigma}$, for all $k \in \{0,1,2\}$.*

Proposition 1.5.2 (Properties of D_σ). *Let $\mathcal{H}$, $\mathcal{G}$ be hypermaps and $\sigma, \tau \in S_{\{0,1,2\}}$. Then:*

1. $\mathrm{D}_1(\mathcal{H}) = \mathcal{H}$; $\mathrm{D}_\tau(\mathrm{D}_\sigma(\mathcal{H})) = \mathrm{D}_{\sigma\tau}(\mathcal{H})$;
2. $\mathcal{H} \to \mathcal{G}$ *if and only if* $\mathrm{D}_\sigma(\mathcal{H}) \to \mathrm{D}_\sigma(\mathcal{G})$; $\mathcal{H} \cong \mathcal{G}$ *if and only if* $\mathrm{D}_\sigma(\mathcal{H}) \cong \mathrm{D}_\sigma(\mathcal{G})$;
3. $\mathcal{H}$ *is* Θ*-conservative if and only if* $\mathrm{D}_\sigma(\mathcal{H})$ *is* $\Theta\sigma$*-conservative;*
4. $\mathcal{H}$ *is* Θ*-uniform if and only if* $\mathrm{D}_\sigma(\mathcal{H})$ *is* $\Theta\sigma$*-uniform;*
5. $\mathcal{H}$ *is* Θ*-regular if and only if* $\mathrm{D}_\sigma(\mathcal{H})$ *is* $\Theta\sigma$*-regular;*
6. $\mathcal{H}$ *and* $\mathrm{D}_\sigma(\mathcal{H})$ *have the same underlying surface;*
7. $\mathrm{Aut}(\mathcal{H}) \cong \mathrm{Aut}(\mathrm{D}_\sigma(\mathcal{H}))$ *and* $\mathrm{Mon}(\mathcal{H}) \cong \mathrm{Mon}(\mathrm{D}_\sigma(\mathcal{H}))$.

As an immediate corollary to Proposition 1.5.2 we get

Corollary 1.5.3. *1. $\mathcal{H}$ is uniform (resp. k-bipartite-uniform) if and only if $\mathrm{D}_\sigma(\mathcal{H})$ is uniform (resp. $k\sigma$-bipartite-uniform);*

2. $\mathcal{H}$ is regular (resp. orientably-regular, k-pseudo-orientably-regular, k-bipartite-regular) if and only if $\mathrm{D}_\sigma(\mathcal{H})$ is regular (resp. orientably-regular, $k\sigma$-pseudo-orientably-regular, $k\sigma$-bipartite-regular);

3. Every k-pseudo-orientably-regular hypermap is uniform.

This result shows that, up to duality, a 2-restrictedly-regular hypermap is orientably-chiral, pseudo-orientably-chiral or bipartite-chiral. Consequently, the classification of all 2-restrictedly-regular hypermaps on a surface $\mathcal{S}$ can be derived from the classification of these 3 types of hypermaps on $\mathcal{S}$.

The 2-skeleton of a convex polyhedron in $\mathbb{R}^3$ can be viewed as a map on the sphere. In particular, the Platonic solids give rise to 5 regular maps on the sphere. For simplicity, we will not differentiate these maps from the corresponding Platonic solids. We denote by $\mathcal{T}$, $\mathcal{C}$, $\mathcal{O}$, $\mathcal{D}$ and $\mathcal{I}$ the tetrahedron, the cube (or hexahedron), the octahedron, the dodecahedron and the icosahedron. These maps have type $(3,2,3)$, $(3,2,4)$, $(4,2,3)$, $(3,2,5)$ and $(5,2,3)$, respectively. It is well-known that if $\mathcal{H}$ is one of these hypermaps and (l,m,n) is the type of $\mathcal{H}$, then $\mathcal{H}$ has hypermap subgroup $\langle (R_1R_2)^l, (R_2R_0)^m, (R_0R_1)^n \rangle^\Delta$, automorphism group $\mathrm{Aut}(\mathcal{H}) \cong \Delta(l,m,n)$, and that $\mathcal{T} \cong \mathrm{D}_{(02)}(\mathcal{T})$, $\mathcal{O} \cong \mathrm{D}_{(02)}(\mathcal{C})$ and $\mathcal{I} \cong \mathrm{D}_{(02)}(\mathcal{D})$. For more information on these hypermaps, see Section 2.1.

Given $k \in \mathbb{N}$, the *dihedral hypermap of order* k, $\mathcal{D}_k$, and the *polygon of order* k, $\mathcal{P}_k$, are the regular hypermaps on the sphere of type $(k,k,1)$ and $(2,2,k)$, and with hypermap subgroup $\langle (R_1R_2)^k, (R_2R_0)^k, R_0R_1 \rangle^\Delta$ and $\langle (R_1R_2)^2, (R_2R_0)^2, (R_0R_1)^k \rangle^\Delta$, respectively. In Figure 1.1 we display $\mathcal{D}_8$ and $\mathcal{P}_4$. The *star hypermap of order* k is the hypermap $\mathcal{S}_k = \mathrm{D}_{(02)}(\mathcal{D}_k)$. The dihedral hypermap of order k has $2k$ flags, 1 vertex, 1 edge and k faces; the polygon of order k has $4k$ flags, k vertices, k edges and 2 faces. Using Corollary 1.4.2 we can see that both $\mathcal{D}_k$ and $\mathcal{P}_k$ are on the sphere. In [15], Breda and Jones denoted the hypermaps $\mathcal{P}_k$ (with k odd) and $\mathrm{D}_{(01)}(\mathcal{D}_k)$ by $\mathcal{D}_k^\ominus$ and $\mathcal{D}_k^*$, respectively; Wilson [73] denoted the hypermap $\mathcal{P}_k$ by ε_k. As remarked in [13], $\mathcal{S}_1 \cong \mathcal{D}_1$, $\mathcal{S}_2$, $\mathcal{P}_1 \cong \mathcal{D}_2$ and $\mathcal{P}_2$ are hypermaps on the sphere with hypermap subgroups Δ^+, $\Delta^{+0\hat{0}}$, $\Delta^{+2\hat{2}}$ and Δ', respectively. In other words, those hypermaps are the hypermaps $\mathcal{T}_{\Delta^+}$, $\mathcal{T}_{\Delta^{+0\hat{0}}}$, $\mathcal{T}_{\Delta^{+2\hat{2}}}$ and $\mathcal{T}_{\Delta'}$.

Coxeter and Moser [33] denoted the regular hypermaps $\mathcal{T}$, $\mathcal{C}$, $\mathcal{O}$, $\mathcal{D}$, $\mathcal{I}$, $\mathcal{P}_{2k}$ and $\mathrm{D}_{(02)}(\mathcal{P}_{2k})$ by $\{3,3\}$, $\{4,3\}$, $\{3,4\}$, $\{5,3\}$, $\{3,5\}$, $\{2k,2\}$ and $\{2,2k\}$, respectively.

$\mathcal{D}_8$ $\mathcal{P}_4$

Figure 1.1: The dihedral hypermap $\mathcal{D}_8$ and the polygon $\mathcal{P}_4$.

A *Petrie polygon* of a hypermap $\mathcal{H}$ is a $\langle R_0R_1R_2\rangle$-orbit on $\Omega_{\mathcal{H}}$. The *length* of a Petrie polygon is its cardinality. Naturally, if $\mathcal{H}$ is regular, all Petrie polygons of $\mathcal{H}$ have the same length. When $\mathcal{M}$ is a map such that all vertices of $\mathcal{M}$ have valency greater than 2, a Petrie polygon of $\mathcal{M}$ is just a 'zig-zag' cycle of edges in which every two consecutive edges belong to a face but no three consecutive edges belong to the same face. Owing to this, the automorphism group of a regular map $\mathcal{M}$ is a transitive permutation representation of the abstract group

$$G^{p,q,r} = \langle a,b,c \mid a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ca)^2 = (abc)^r = 1\rangle$$

defined by Coxeter in [29]. It is well-known that $G^{3,3,4} \cong S_4$, $G^{3,4,6} \cong S_4 \times C_2$, $G^{3,5,5} \cong A_5$, $G^{3,5,10} \cong A_5 \times C_2$ and $G^{2,k,2k} \cong D_k \times C_2$ (see Table 1 of [30], for instance). When k is even, $G^{2,k,k} \cong G^{2,k,2k} \cong D_k \times C_2$.

1.6 Constructing bipartite hypermaps

By the Reidemeister-Schreier rewriting process [42] it can be shown that

$$\Delta^{\hat{0}} \cong C_2 * C_2 * C_2 * C_2 = \langle R_1\rangle * \langle R_2\rangle * \langle {R_1}^{R_0}\rangle * \langle {R_2}^{R_0}\rangle.$$

As a consequence we have several epimorphisms from $\Delta^{\hat{0}}$ to Δ.

Let $\varphi : \Delta^{\hat{0}} \to \Delta$ be an epimorphism. Then, if H is a subgroup of Δ and $g \in \Delta$, then $H\varphi^{-1}$ and $H^g\varphi^{-1}$ are conjugate subgroups of $\Delta^{\hat{0}}$ and hence conjugate subgroups of Δ. After all, if $g = d\varphi$, then $H^g\varphi^{-1} = H^{d\varphi}\varphi^{-1} = (H\varphi^{-1})^d$. In other words, given an epimorphism $\varphi : \Delta^{\hat{0}} \to \Delta$ and a hypermap $\mathcal{H}$ with hypermap subgroup H we can construct another hypermap $\mathcal{H}^{\varphi^{-1}}$ with hypermap subgroup $H\varphi^{-1}$.

$$\mathcal{H}^{\varphi^{-1}} \left\{ \begin{array}{ccc} \Delta & & \\ \|{}^{2} & & \\ \Delta^{\hat{0}} & \xrightarrow{\varphi} & \Delta \\ | & & | \\ H\varphi^{-1} & \longrightarrow & H \end{array} \right\} \mathcal{H}$$

Lemma 1.6.1. *Let $\varphi : \Delta^{\hat{0}} \to \Delta$ be an epimorphism and $\mathcal{H}$ and $\mathcal{G}$ hypermaps. Then:*

1. *$\mathcal{H}^{\varphi^{-1}}$ is bipartite hypermap with twice the number of flags of $\mathcal{H}$.*
2. *$\mathcal{H}^{\varphi^{-1}}$ is bipartite-regular if and only if $\mathcal{H}$ is regular.*
3. *If $\mathcal{H}$ covers $\mathcal{G}$, then $\mathcal{H}^{\varphi^{-1}}$ covers $\mathcal{G}^{\varphi^{-1}}$.*
4. *If $\mathcal{H}$ is isomorphic to $\mathcal{G}$, then $\mathcal{H}^{\varphi^{-1}}$ is isomorphic to $\mathcal{G}^{\varphi^{-1}}$.*

Proof. Let H and G be hypermap subgroups of $\mathcal{H}$ and $\mathcal{G}$.
1. Clearly, $H\varphi^{-1} \leq \Delta\varphi^{-1} = \Delta^{\hat{0}}$ and hence $\mathcal{H}^{\varphi^{-1}}$ is $\Delta^{\hat{0}}$-conservative, that is, bipartite. By Proposition A.1.1, $[\Delta^{\hat{0}} : H\varphi^{-1}] = [\Delta : H]$ and hence

$$|\Omega_{\mathcal{H}^{\varphi^{-1}}}| = [\Delta : H\varphi^{-1}] = [\Delta : \Delta^{\hat{0}}][\Delta^{\hat{0}} : H\varphi^{-1}] = 2[\Delta : H] = 2|\Omega_{\mathcal{H}}|.$$

2. When φ is onto, $\mathcal{H}^{\varphi^{-1}}$ is bipartite-regular $\Leftrightarrow H\varphi^{-1} \lhd \Delta^{\hat{0}} \Leftrightarrow H \lhd \Delta \Leftrightarrow \mathcal{H}$ is regular.
3. If $H \subseteq G^g$ and $g = d\varphi$, for some $d \in \Delta^{\hat{0}}$, then $H\varphi^{-1} \subseteq G^g\varphi^{-1} = G^{d\varphi}\varphi^{-1} = (G\varphi^{-1})^d$.
4. Follows from 3. □

Among many possible canonical epimorphisms $\varphi : \Delta^{\hat{0}} \to \Delta$, there are two, φ_W and φ_P, defined by

$$R_1\varphi_W = R_1, \quad R_2\varphi_W = R_2, \quad {R_1}^{R_0}\varphi_W = R_0, \quad {R_2}^{R_0}\varphi_W = R_2, \tag{1.6}$$
$$R_1\varphi_P = R_1, \quad R_2\varphi_P = R_2, \quad {R_1}^{R_0}\varphi_P = R_0, \quad {R_2}^{R_0}\varphi_P = R_0, \tag{1.7}$$

that induce very interesting constructions. The first construction gives rise to the correspondence between hypermaps and biparte maps described by Walsh in [67]. We denote by Walsh($\mathcal{H}$) the hypermap $\mathcal{H}^{\varphi_W^{-1}}$ and by Pin($\mathcal{H}$) the hypermap $\mathcal{H}^{\varphi_P^{-1}}$. In Figure 1.2 we illustrate these 2 constructions.

Lemma 1.6.2. $\ker\varphi_W = \langle R_2{R_2}^{R_0}\rangle^{\Delta^{\hat{0}}} = \langle R_2{R_2}^{R_0}\rangle^{\Delta}$ *and* $\ker\varphi_P = \langle {R_1}^{R_0}{R_2}^{R_0}\rangle^{\Delta^{\hat{0}}}$.

Let $\psi : \Delta \to \Delta^{\hat{0}}$ be the group homomorphism defined by

$$R_1\psi = R_1, \quad R_2\psi = R_2, \quad R_0\psi = {R_1}^{R_0}. \tag{1.8}$$

Since $R_i\psi\varphi_W = R_i = R_i\psi\varphi_P$, for every $i \in \{0,1,2\}$, $\psi\varphi_W = 1_\Delta = \psi\varphi_P$ and ψ is injective.

Proposition 1.6.3. *Let φ be φ_W or φ_P. Then $\Delta^+\varphi^{-1} = \Delta^+ \cap \Delta^{\hat{0}}$.*

Proof. We use induction on $\Delta^{\hat{0}} = \langle R_1, R_2, {R_1}^{R_0}, {R_2}^{R_0}\rangle$ to prove that for all $g \in \Delta^{\hat{0}}$, $g \in \Delta^+\varphi^{-1}$ if and only if $g \in \Delta^+$.

Let $S = \{g \in \Delta^{\hat{0}} \mid g \in \Delta^+\varphi^{-1} \Leftrightarrow g \in \Delta^+\}$. Then:
(a) $R_1, R_2, {R_1}^{R_0}, {R_2}^{R_0} \in S$, because $R_1, R_2, {R_1}^{R_0}, {R_2}^{R_0}, R_1\varphi, R_2\varphi, ({R_1}^{R_0})\varphi, ({R_2}^{R_0})\varphi \notin \Delta^+$.
(b) For all $g_1, g_2 \in S$,

$$\begin{aligned}
g_1g_2 \in \Delta^+\varphi^{-1} \quad &\Leftrightarrow \quad (g_1g_2)\varphi = g_1\varphi g_2\varphi \in \Delta^+ \\
&\Leftrightarrow \quad g_1\varphi, g_2\varphi \in \Delta^+ \text{ or } g_1\varphi, g_2\varphi \notin \Delta^+ \\
&\Leftrightarrow \quad g_1, g_2 \in \Delta^+\varphi^{-1} \text{ or } g_1, g_2 \notin \Delta^+\varphi^{-1} \\
&\Leftrightarrow \quad g_1, g_2 \in \Delta^+ \text{ or } g_1, g_2 \notin \Delta^+ \\
&\Leftrightarrow \quad g_1g_2 \in \Delta^+,
\end{aligned}$$

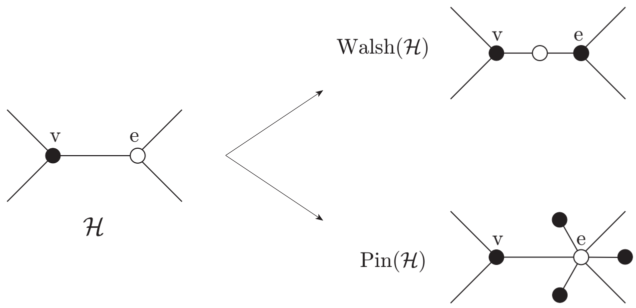

Figure 1.2: Topological construction of Walsh($\mathcal{H}$) and Pin($\mathcal{H}$).

that is, $g_1g_2 \in S$.
(c) For all $g \in S$, $g^{-1} \in \Delta^+\varphi^{-1} \Leftrightarrow g \in \Delta^+\varphi^{-1} \Leftrightarrow g \in \Delta^+ \Leftrightarrow g^{-1} \in \Delta^+$, that is $g^{-1} \in S$.
By induction on $\Delta^{\hat{0}}$, we have $S = \Delta^{\hat{0}}$. Thus $\Delta^+\varphi^{-1} = \Delta^+\varphi^{-1} \cap \Delta^{\hat{0}} = \Delta^+ \cap \Delta^{\hat{0}}$. □

1.6.1 The Walsh construction

Let $W := H{\varphi_W}^{-1}$ be a hypermap subgroup of $\mathcal{W} :=$ Walsh($\mathcal{H}$). By Lemma 1.6.1, $\mathcal{W}$ is a bipartite hypermap with twice the number of flags of $\mathcal{H}$ which is bipartite-regular if and only if $\mathcal{H}$ is regular. In addition, the Walsh construction has the following properties.

Theorem 1.6.4 (Properties of Walsh). *Let $\mathcal{H}$ be a hypermap and $\mathcal{W} :=$ Walsh($\mathcal{H}$). Then:*

1. *$\mathcal{W}$ is a map;*
2. *$\mathcal{W}$ is orientable if and only if $\mathcal{H}$ is orientable;*
3. *$V(\mathcal{W}) = V(\mathcal{H}) + E(\mathcal{H})$, $E(\mathcal{W}) = |\Omega_{\mathcal{H}}|/2$, $F(\mathcal{W}) = F(\mathcal{H})$;*
4. *$\mathcal{W}$ has the same underlying surface as $\mathcal{H}$;*
5. *$\mathcal{H}$ is uniform of type (l, m, n) if and only if $\mathcal{W}$ is bipartite-uniform of bipartite-type $(l, m; 2; 2n)$.*

Proof. 1. We claim that $(R_2R_0)^2 \in W^g$, for all $g \in \Delta$. Indeed, if $g \in \Delta^{\hat{0}}$, then

$$[(R_2R_0)^2]^{g^{-1}}\varphi_W = [(R_2R_0)^2\varphi_W]^{g^{-1}\varphi_W} = 1 \in H,$$

so $(R_2R_0)^2 \in W^g$. Else, if $g \notin \Delta^{\hat{0}}$, then $gR_0 \in \Delta^{\hat{0}}$ and

$$[(R_2R_0)^2]^{g^{-1}}\varphi_W = [(R_0R_2)^2]^{R_0g^{-1}}\varphi_W = [(R_0R_2)^2\varphi_W]^{(gR_0)^{-1}\varphi_W} = 1 \in H,$$

that is, $(R_2R_0)^2 \in W^g$. To put it another way, all edges have, at most, valency 2. On the other hand, since $W \subseteq \Delta^{\hat{0}}$, $\mathcal{W}$ is bipartite so, by Lemma 1.3.6, all edges have even valencies. Consequently, all edges of $\mathcal{W}$ have valency 2 and $\mathcal{W}$ is a map.
2. Follows from Proposition 1.6.3.
3. The mapping

$$\begin{array}{ccc} \{Wg \mid g \in \Delta^{\hat{0}}\} & \longrightarrow & \Delta/_r H \\ Wg & \longmapsto & H(g\varphi_W) \end{array}$$

is a bijection between the $\Delta^{\hat{0}}$-orbit of $\Omega_{\mathcal{W}}$ containing W and $\Omega_{\mathcal{H}}$. Since $R_1\varphi_W = R_1$ and $R_2\varphi_W = R_2$, the flags Wg and Wg' are in the same vertex of $\mathcal{W}$ if and only if $H(g\varphi_W)$ and $H(g'\varphi_W)$ are in the same vertex of $\mathcal{H}$. Consequently, there is a bijective correspondence between the set of vertices of $\mathcal{W}$ in the $\Delta^{\hat{0}}$-orbit containing the flag W and the set of vertices of $\mathcal{H}$. Similarly, since $R_1{}^{R_0}\varphi_W = R_0$ and $R_2{}^{R_0}\varphi_W = R_2$ the mapping

$$\begin{array}{ccc} \{Wg \mid g \in \Delta^{\hat{0}}\} & \longrightarrow & \Delta/_r H \\ Wg & \longmapsto & H(g\varphi_W) \end{array}$$

induces a bijective correspondence between the set of vertices of $\mathcal{W}$ in the $\Delta^{\hat{0}}$-orbit containing the flag WR_0 and the set of edges of $\mathcal{H}$. Owing to this, the number of vertices of $\mathcal{W}$, $V(\mathcal{W})$, is equal to the sum of the number of the vertices of $\mathcal{H}$, $V(\mathcal{H})$, with the number of edges of $\mathcal{H}$, $E(\mathcal{H})$.

We already know that $\mathcal{W}$ is a map, so every edge has valency 2 and the number of edges of $\mathcal{W}$, $E(\mathcal{W})$, is $|\Omega_{\mathcal{W}}|/4 = |\Omega_{\mathcal{H}}|/2$ (see Lemma 1.6.1).

Because $R_1{}^{R_0}\varphi_W = R_0$ and $R_1\varphi_W = R_1$, the face of $\mathcal{W}$ containing the flag Wg also contains WgR_0 and has twice the cardinality of the flag of $\mathcal{H}$ containing $H(g\varphi)$, if $g \in \Delta^{\hat{0}}$, or $H(g\varphi)$, otherwise. Thus, $\mathcal{W}$ and $\mathcal{H}$ have the same number of faces.

4. Using Lemma 1.2,

$$\begin{aligned} \chi_{\mathcal{W}} &= V(\mathcal{W}) + E(\mathcal{W}) + F(\mathcal{W}) - \frac{|\Omega_{\mathcal{W}}|}{2} \\ &= (V(\mathcal{H}) + E(\mathcal{H})) + \frac{|\Omega_{\mathcal{H}}|}{2} + F(\mathcal{H}) - |\Omega_{\mathcal{H}}| \\ &= V(\mathcal{H}) + E(\mathcal{H}) + F(\mathcal{H}) - \frac{|\Omega_{\mathcal{H}}|}{2} \\ &= \chi_{\mathcal{H}}. \end{aligned}$$

5. Follows from the proof of 3. □

Theorem 1.6.5. *The hypermap $\mathcal{H}$ is a bipartite map if and only if $\mathcal{H} \cong \mathrm{Walsh}(\mathcal{G})$, for some hypermap $\mathcal{G}$. Moreover, $\mathcal{H}$ is bipartite-uniform of type $(l, m; 2; 2n)$ if and only if $\mathcal{G}$ is uniform of type (l, m, n); $\mathcal{H}$ is bipartite-regular of type $(l, m; 2; 2n)$ if and only if $\mathcal{G}$ is regular of type (l, m, n).*

Proof. Only the necessary condition needs to be proved. If $\mathcal{H}$ is bipartite, then $H \subseteq \Delta^{\hat{0}}$. Since $\mathcal{H}$ is a map, $((R_2R_0)^2)^g \in H$ for all $g \in \Delta$, so $\ker\varphi_W = \langle (R_2R_0)^2\rangle^{\Delta} \subseteq H$. Because of this, $H\varphi_W{\varphi_W}^{-1} = H\ker\varphi_W = H$ and hence $\mathcal{H} \cong \mathrm{Walsh}(\mathcal{G})$ where $\mathcal{G}$ is the hypermap with hypermap subgroup $G = H\varphi_W$. □

Theorem 1.6.6. *1.* $\mathrm{Walsh}(\mathrm{D}_{(01)}(\mathcal{H})) \cong \mathrm{Walsh}(\mathcal{H})$.

2. If $\mathrm{Walsh}(\mathcal{H}) \cong \mathrm{Walsh}(\mathcal{G})$, *then $\mathcal{H} \cong \mathcal{G}$ or $\mathcal{H} \cong \mathrm{D}_{(01)}(\mathcal{G})$.*

Proof. If H is a hypermap subgroup of $\mathcal{H}$, then $H{\varphi_W}^{-1}$ and $H\overline{(01)}{\varphi_W}^{-1}$ are hypermap subgroups of $\mathrm{Walsh}(\mathcal{H})$ and $\mathrm{Walsh}(\mathrm{D}_{(01)}(\mathcal{H}))$, respectively. Since

$$\begin{array}{ll} R_1\varphi_W\overline{(01)} = R_0 = R_1{}^{R_0}\varphi_W, & R_1{}^{R_0}\varphi_W\overline{(01)} = R_1 = R_1\varphi_W, \\ R_2\varphi_W\overline{(01)} = R_2 = R_2{}^{R_0}\varphi_W, & R_2{}^{R_0}\varphi_W\overline{(01)} = R_2 = R_2\varphi_W, \end{array}$$

we have that $g\varphi_W\overline{(01)} = g^{R_0}\varphi_W$, for all $g \in \Delta^{\hat{0}}$.
1. Since

$$g \in H\overline{(01)}{\varphi_W}^{-1} \Leftrightarrow g\varphi_W\overline{(01)} \in H \Leftrightarrow g^{R_0}\varphi_W \in H \Leftrightarrow g \in (H{\varphi_W}^{-1})^{R_0},$$

$H\overline{(01)}{\varphi_W}^{-1} = (H{\varphi_W}^{-1})^{R_0}$. Hence, Walsh($\mathrm{D}_{(01)}(\mathcal{H})$) and Walsh($\mathcal{H}$) are isomorphic.
2. Let H and G be hypermap subgroups of $\mathcal{H}$ and $\mathcal{G}$. Assume that Walsh($\mathcal{H}$) $\cong$ Walsh($\mathcal{G}$). Then $H{\varphi_W}^{-1} = (G{\varphi_W}^{-1})^g$, for some $g \in \Delta$. If $g \in \Delta^{\hat{0}}$, then

$$H = H{\varphi_W}^{-1}\varphi_W = (G{\varphi_W}^{-1})^g\varphi_W = (G{\varphi_W}^{-1}\varphi_W)^{g\varphi_W} = G^{g\varphi_W}$$

and $\mathcal{H} \cong \mathcal{G}$; else if $g \notin \Delta^{\hat{0}}$, then $R_0 g \in \Delta^{\hat{0}}$,

$$\begin{aligned} H &= H{\varphi_W}^{-1}\varphi_W = (G{\varphi_W}^{-1})^g\varphi_W = (G{\varphi_W}^{-1})^{R_0R_0g}\varphi_W \\ &= [(G{\varphi_W}^{-1})^{R_0}\varphi_W]^{R_0g\varphi_W} = (G\overline{(01)}{\varphi_W}^{-1}\varphi_W)^{R_0g\varphi_W} = (G\overline{(01)})^{R_0g\varphi_W}, \end{aligned}$$

and $\mathcal{H} \cong D_{(01)}(\mathcal{G})$. □

Remark 1.6.7. Walsh($\mathcal{D}_k$) $\cong \mathrm{D}_{(02)}(\mathcal{P}_k)$ and Walsh($\mathcal{P}_k$) $\cong \mathcal{P}_{2k}$, for all $k \in \mathbb{N}$.

Given a hypermap $\mathcal{H}$, we can construct a map $\mathcal{M}$, called the *medial map* of $\mathcal{H}$, in the following way. The set of vertices of $\mathcal{M}$ is the set of edges of $\mathcal{H}$, and two vertices of $\mathcal{M}$ are connected by an edge if and only if the corresponding edges of $\mathcal{H}$ are both incident to a common vertex v of $\mathcal{H}$ and a common face f of $\mathcal{H}$. The set of faces of $\mathcal{M}$ corresponds in a natural way to the union of the sets of vertices and faces of $\mathcal{H}$. This construction is an extension of the well-known medial map of a map. We denote the medial map of $\mathcal{H}$ by Med($\mathcal{H}$). One can easily see that Med($\mathcal{H}$) is a face-bipartite hypermap such that Med($\mathcal{H}$) $= \mathrm{D}_{(02)}$(Walsh($\mathrm{D}_{(12)}(\mathcal{H})$)).

1.6.2 The Pin construction

Let $P := H{\varphi_P}^{-1}$ be a hypermap subgroup of $\mathcal{P} :=$ Pin($\mathcal{H}$). Like in the previous construction, Lemma 1.6.1 ensures that $\mathcal{P}$ is a bipartite hypermap with twice the number of flags of $\mathcal{H}$ and that $\mathcal{P}$ is bipartite-regular if and only if $\mathcal{H}$ is regular. In addition, the Pin construction has the following properties.

Theorem 1.6.8 (Properties of Pin)**.** *Let $\mathcal{H}$ be a hypermap and $\mathcal{P} :=$ Pin($\mathcal{H}$). Then:*

1. *All vertices in one $\Delta^{\hat{0}}$-orbit of $\mathcal{P}$ have valency 1;*
2. *$\mathcal{P}$ is orientable if and only if $\mathcal{H}$ is orientable,*
3. *$V(\mathcal{P}) = V(\mathcal{H}) + |\Omega_{\mathcal{H}}|/2$, $E(\mathcal{P}) = E(\mathcal{H})$, $F(\mathcal{P}) = F(\mathcal{H})$;*
4. *$\mathcal{P}$ has the same underlying surface as $\mathcal{H}$,*
5. *$\mathcal{H}$ is uniform of type (l, m, n) if and only if $\mathcal{P}$ is bipartite-uniform of bipartite-type $(1, l; 2m; 2n)$.*

Proof. 1. We claim that all vertices in the same $\Delta^{\hat{0}}$-orbit of the vertex containing the flag PR_0 have valency 1. To prove this, we only need to show that $R_1R_2 \in P^{R_0 g}$, for all $g \in \Delta^{\hat{0}}$. Given $g \in \Delta^{\hat{0}}$, g^{R_0} also belongs to $\Delta^{\hat{0}}$ and

$$(R_1R_2)^{g^{-1}R_0}\varphi_P = ((R_1R_2)^{R_0})^{(g^{R_0})^{-1}}\varphi_P = ((R_1R_2)^{R_0}\varphi_P)^{(g^{R_0})^{-1}\varphi_P} = 1 \in H,$$

and hence $R_1R_2 \in (H{\varphi_P}^{-1})^{R_0 g} = P^{R_0 g}$, for all $g \in \Delta^{\hat{0}}$.
2. Follows from Proposition 1.6.3.
3. Similar to the proof of 3 of 1.6.4.
4. Using Lemma 1.2,

$$\begin{aligned}
\chi_{\mathcal{P}} &= V(\mathcal{P}) + E(\mathcal{P}) + F(\mathcal{P}) - \frac{|\Omega_{\mathcal{P}}|}{2} \\
&= \left(V(\mathcal{H}) + \frac{|\Omega_{\mathcal{H}}|}{2}\right) + E(\mathcal{H}) + F(\mathcal{H}) - |\Omega_{\mathcal{H}}| \\
&= V(\mathcal{H}) + E(\mathcal{H}) + F(\mathcal{H}) - \frac{|\Omega_{\mathcal{H}}|}{2} \\
&= \chi_{\mathcal{H}}.
\end{aligned}$$

5. Similar to the proof of 5 of 1.6.4. □

Theorem 1.6.9. *The hypermap $\mathcal{H}$ is a bipartite hypermap such that all vertices in one $\Delta^{\hat{0}}$-orbit have valency* 1 *if and only if $\mathcal{H} \cong \mathrm{Pin}(\mathcal{G})$, for some hypermap $\mathcal{G}$. Moreover, $\mathcal{H}$ is bipartite-uniform of type $(1, l; 2m; 2n)$ if and only if $\mathcal{G}$ is uniform of type (l, m, n); $\mathcal{H}$ is bipartite-regular of type $(1, l; 2m; 2n)$ if and only if $\mathcal{G}$ is regular of type (l, m, n).*

Proof. As in Theorem 1.6.5, only the necessary condition needs to be proved. Let H be a hypermap subgroup of $\mathcal{H}$. By taking H^{R_0} instead of H if necessary, we may assume, without loss of generality, that all vertices in the $\Delta^{\hat{0}}$-orbit of the vertex that contains the flag HR_0 have valency 1, i.e, $HR_0gR_1R_2 = HR_0g$ for all $g \in \Delta^{\hat{0}}$. Then $\ker\varphi_P \subseteq H$, so $H\varphi_P{\varphi_P}^{-1} = H\ker\varphi_P = H$ and $\mathcal{H}$ is isomorphic to $\mathrm{Pin}(\mathcal{G})$, where $\mathcal{G}$ is the hypermap with hypermap subgroup $G = H\varphi_P$. □

Theorem 1.6.10. *1.* $\mathrm{Pin}(\mathrm{D}_{(12)}(\mathcal{H})) \cong \mathrm{D}_{(12)}(\mathrm{Pin}(\mathcal{H}))$.

2. If $\mathrm{Pin}(\mathcal{H}) \cong \mathrm{Pin}(\mathcal{G})$, *then* $\mathcal{H} \cong \mathcal{G}$.

Proof. Let H be a hypermap subgroup of $\mathcal{H}$. Then $H\overline{(12)}{\varphi_P}^{-1}$ and $H{\varphi_P}^{-1}\overline{(12)}$ are hypermap subgroups of $\mathrm{Pin}(\mathrm{D}_{(12)}(\mathcal{H}))$ and $\mathrm{D}_{(12)}(\mathrm{Pin}(\mathcal{H}))$, respectively. Since

$$\begin{array}{ll}
(R_1\overline{(12)})\varphi_P = R_2 = R_1\varphi_P\overline{(12)}, & ({R_1}^{R_0}\overline{(12)})\varphi_P = R_0 = {R_1}^{R_0}\varphi_P\overline{(12)}, \\
(R_2\overline{(12)})\varphi_P = R_1 = R_2\varphi_P\overline{(12)}, & ({R_2}^{R_0}\overline{(12)})\varphi_P = R_0 = {R_2}^{R_0}\varphi_P\overline{(12)},
\end{array}$$

we have $g\overline{(12)}\varphi_P = g\varphi_P\overline{(12)}$, for all $g \in \Delta^{\hat{0}}$.
1. Since

$$g \in H{\varphi_P}^{-1}\overline{(12)} \Leftrightarrow g\overline{(12)}\varphi_P \in H \Leftrightarrow g\varphi_P\overline{(12)} \in H \Leftrightarrow g \in H\overline{(12)}{\varphi_P}^{-1},$$

$H{\varphi_P}^{-1}\overline{(12)} = H\overline{(12)}{\varphi_P}^{-1}$ and hence $\mathrm{Pin}(\mathrm{D}_{(12)}(\mathcal{H}))$ and $\mathrm{D}_{(12)}(\mathrm{Pin}(\mathcal{H}))$ are isomorphic.
2. Let H and G be hypermap subgroups of $\mathcal{H}$ and $\mathcal{G}$. Assume that $\mathrm{Pin}(\mathcal{H}) \cong \mathrm{Pin}(\mathcal{G})$. Then $H{\varphi_P}^{-1} = (G{\varphi_P}^{-1})^g$ for some $g \in \Delta$. If $g \in \Delta^{\hat{0}}$, then

$$H = H{\varphi_P}^{-1}\varphi_P = (G{\varphi_P}^{-1})^g\varphi_P = (G{\varphi_P}^{-1}\varphi_P)^{g\varphi_P} = G^{g\varphi_P}$$

and $\mathcal{H} \cong \mathcal{G}$.

Now assume that $g \notin \Delta^{\hat{0}}$. We claim that all vertices of $\mathcal{H}$ have valency 1. Given $d \in \Delta$, let $a \in \Delta^{\hat{0}}$ such that $d = a\varphi$. Then $R_0(ga)^{-1} \in \Delta^{\hat{0}}$ and

$$(R_1R_2)^{(ga)^{-1}}\varphi = (R_1R_2)^{R_0R_0(ga)^{-1}}\varphi = (R_1R_2)^{R_0}\varphi^{R_0(ga)^{-1}\varphi} = 1 \in G,$$

that is, $R_1R_2 \in (G\varphi^{-1})^{ga} = (H\varphi^{-1})^a = H^{a\varphi}\varphi^{-1} = H^d\varphi^{-1}$. In addition, $R_1R_2 = (R_1R_2)\varphi \in H^d\varphi^{-1}\varphi = H^d$. Thus, the vertex of $\mathcal{H}$ containing Hd has valency 1. By Lemma 1.4.4, $\mathcal{H} \cong \mathcal{S}_k$. Similarly, one can see that $\mathcal{G} \cong \mathcal{S}_k \cong \mathcal{H}$. □

Remark 1.6.11. $\mathrm{Pin}(\mathcal{S}_k) \cong \mathcal{S}_{2k}$, for all $k \in \mathbb{N}$.

1.7 The operator Orient

In this section we see how to obtain non-orientable hypermaps from orientable hypermaps having an involutory orientation-reversing automorphism which is not a reflection.

Given a hypermap $\mathcal{H}$ with hypermap subgroup H, let $\mathrm{Orient}(\mathcal{H})$ be the hypermap with hypermap subgroup $H \cap \Delta^+$. Then $\mathrm{Orient}(\mathcal{H})$ is the smallest orientable hypermap covering $\mathcal{H}$. When $\mathcal{H}$ is orientable, $\mathrm{Orient}(\mathcal{H})$ is isomorphic to $\mathcal{H}$. Otherwise, $\mathrm{Orient}(\mathcal{H})$ is the disjoint product $\mathcal{H} \times \mathcal{D}_1$ of Breda and Jones [13], an extension to hypermaps of Wilson's parallel product of maps [74]. Following [15], we also denote $\mathrm{Orient}(\mathcal{H})$ by $\mathcal{H}^+$ and $H \cap \Delta^+$ by H^+.

Theorem 1.7.1 (Properties of Orient)**.** *Let $\mathcal{H}$ be a non-orientable hypermap with hypermap subgroup H, $\mathcal{H}^+ = \mathrm{Orient}(\mathcal{H})$, $H^+ = H \cap \Delta^+$ and $\Theta \lhd \Delta$.*

1. *$[H : H^+] = 2$, so $H = H^+ \cup H^+g$, for some $g \in H$; since our hypermaps have no boundary, g cannot be a conjugate of R_0, R_1 or R_2.*

2. *$|\Omega_{\mathcal{H}^+}| = 2|\Omega_{\mathcal{H}}|$;*

3. *the k-face of $\mathcal{H}$ containing Ha and the k-faces of $\mathcal{H}^+$ containing H^+a and H^+ga have the same valency, for all $a \in \Delta$, and for all $k \in \{0, 1, 2\}$.*

4. *$V(\mathcal{H}^+) = 2V(\mathcal{H})$, $E(\mathcal{H}^+) = 2E(\mathcal{H})$, $F(\mathcal{H}^+) = 2F(\mathcal{H})$;*

5. *$\chi_{\mathcal{H}^+} = 2\chi_{\mathcal{H}}$, $g_{\mathcal{H}^+} = g_{\mathcal{H}} - 1$;*

6. *the covering $\pi : \Delta/_r H^+ \to \Delta/_r H$, $H^+a \mapsto Ha$, the automorphism of $\mathcal{H}^+$, $\varphi : \Delta/_r H^+ \to \Delta/_r H^+$, $H^+a \mapsto H^+ga$, and the identity automorphism of $\mathcal{H}$, 1, comute according to the following diagram:*

$$\begin{tikzcd} \Delta/_r H^+ \arrow[r, "\varphi"] \arrow[d, "\pi"'] & \Delta/_r H^+ \arrow[d, "\pi"] \\ \Delta/_r H \arrow[r, "1"] & \Delta/_r H \end{tikzcd}$$

   that is, there is an involutory orientation-reversing automorphism φ of $\mathcal{H}^+$ which is not a reflection such that $\varphi \circ \pi = \pi$.

7. *If $\mathcal{H}$ is Θ-conservative, then $\mathcal{H}^+$ is also Θ-conservative;*

8. *If $\mathcal{H}$ is Θ-uniform, then $\mathcal{H}^+$ is also Θ-uniform;*

9. *If $\mathcal{H}$ is Θ-regular, then $\mathcal{H}^+$ is also Θ-regular;*

10. *For all $\sigma \in S_{\{0,1,2\}}$, $\mathrm{D}_\sigma(\mathcal{H}^+) \cong (\mathrm{D}_\sigma(\mathcal{H}))^+$.*

Proof. 1. Follows from Corollary A.1.2.
2. By 1, $|\Omega_{\mathcal{H}^+}| = [\Delta : H^+] = [\Delta : H] \cdot [H : H^+] = 2[\Delta : H] = 2|\Omega_{\mathcal{H}}|$.
3. If $d \in \Delta^+$, then $d \in H^+ \Leftrightarrow d \in H = H^g \Leftrightarrow d \in H^+$. More generally, if $d \in \Delta^+$, then $d \in (H^+)^a = (H^a)^+ \Leftrightarrow d \in H^a = H^{ga} \Leftrightarrow d \in (H^+)^a = (H^a)^+$. The result follows by taking d as $(R_1R_2)^p$, $(R_2R_0)^q$ and $(R_0R_1)^r$.
4. Follows from 2 and 3.
5. Follows from 4. Using Lemma 1.4.1,

$$\begin{aligned} \chi_{\mathcal{H}^+} &= V(\mathcal{H}^+) + E(\mathcal{H}^+) + F(\mathcal{H}^+) - \frac{|\Omega_{\mathcal{H}^+}|}{2} \\ &= 2V(\mathcal{H}) + 2E(\mathcal{H}) + 2F(\mathcal{H}) - |\Omega_{\mathcal{H}}| \\ &= 2\chi_{\mathcal{H}}. \end{aligned}$$

Since $\mathcal{H}^+$ is orientable but $\mathcal{H}$ is not, $\chi_{\mathcal{H}^+} = 2 - 2g_{\mathcal{H}^+}$ and $\chi_{\mathcal{H}} = 2 - g_{\mathcal{H}}$, so

$$g_{\mathcal{H}^+} = \frac{2 - \chi_{\mathcal{H}^+}}{2} = \frac{2 - 2\chi_{\mathcal{H}}}{2} = 1 - \chi_{\mathcal{H}} = g_{\mathcal{H}} - 1.$$

6. For every $a \in \Delta$, $(H^+a)\varphi\pi = (H^+ga)\pi = Hga = Ha = (H^+a)\pi$, because $g \in H$.
7. Because $H^+ \subseteq H$.
8. Follows from 7 and 3.
9. By 7 and because

$$\mathrm{N}_\Delta(H^+) = \mathrm{N}_\Delta(H \cap \Delta^+) \supseteq \mathrm{N}_\Delta(H) \cap \mathrm{N}_\Delta(\Delta^+) = \mathrm{N}_\Delta(H) \cap \Delta = \mathrm{N}_\Delta(H).$$

10. By Lemma 1.5.1, $\Delta^+\overline{\sigma} = \Delta^+$. Since $\overline{\sigma}$ is bijective,

$$H^+\overline{\sigma} = (H \cap \Delta^+)\overline{\sigma} = H\overline{\sigma} \cap \Delta^+\overline{\sigma} = H\overline{\sigma} \cap \Delta^+ = (H\overline{\sigma})^+,$$

and hence $\mathrm{D}_\sigma(\mathcal{H}^+)$ and $(\mathrm{D}_\sigma(\mathcal{H}))^+$ are isomorphic. □

When $\mathcal{H}$ is non-orientable, $\mathrm{Orient}(\mathcal{H})$ is called the *orientable double cover* of $\mathcal{H}$. A hypermap $\mathcal{K}$ is called *antipodal* (see [53, 22] for maps) if $\mathcal{K}$ is the orientable double cover of a non-orientable hypermap $\mathcal{H}$.

Corollary 1.7.2. *If $\mathcal{H}$ is a Θ-regular hypermap on a non-orientable surface of genus $g_{\mathcal{H}}$, then $\mathcal{H}^+ = \mathrm{Orient}(\mathcal{H})$ is a Θ-regular hypermap on an orientable surface of genus $g_{\mathcal{H}^+} = g_{\mathcal{H}} - 1$, with twice the numbers of flags, vertices, edges and faces of $\mathcal{H}$, and having an involutory Θ-conservative orientation-reversing automorphism φ which is not a reflection.*

In particular all Θ-regular hypermaps on the projective plane and on the Klein bottle are obtained from Θ-regular hypermaps on the sphere and on the torus, respectively. In Chapter 3 we can find examples showing that, in general, the converses of 7, 8 and 9 of Theorem 1.7.1 are not true.

Corollary 1.7.3. *If $\mathcal{H}$ is regular, then $\mathcal{H}^+ = \mathrm{Orient}(\mathcal{H})$ is also regular and the center of* $\mathrm{Aut}(\mathcal{H}^+)$ *is non-trivial, that is* $|\,\mathrm{Z}(\mathrm{Aut}(\mathcal{H}^+))| \geq 2$.

Proof. When $\mathcal{H}$ is regular, $\mathcal{H}^+$ is also regular, $\mathrm{Aut}(\mathcal{H}) \cong \Delta/H$ and $\mathrm{Aut}(\mathcal{H}^+) \cong \Delta/H^+$. Since $H^+ \subseteq H$, the mapping $\varphi : \Delta/H^+ \to \Delta/H$, $H^+g \mapsto Hg$ is an epimorphism, and $\ker\varphi$, being a normal subgroup of Δ/H^+ with 2 elements, is contained in $\mathrm{Z}(\Delta/H^+)$. □

Now we show that Orient commutes with Walsh and Pin.

Proposition 1.7.4. *Let $\varphi : \Delta^{\hat{0}} \to \Delta$ be an epimorphism such that $\Delta^+\varphi^{-1} = \Delta^+ \cap \Delta^{\hat{0}}$. Then $(\mathcal{H}^+)^{\varphi^{-1}}$ is isomorphic to $(\mathcal{H}^{\varphi^{-1}})^+$.*

Proof. By Proposition 1.6.3, $(H^+)\varphi^{-1}$ and $(H\varphi^{-1})^+$ are hypermap subgroups of $(\mathcal{H}^+)^{\varphi^{-1}}$ and $(\mathcal{H}^{\varphi^{-1}})^+$, respectively. Since

$$\begin{aligned}(H^+)\varphi^{-1} &= (H \cap \Delta^+)\varphi^{-1} = H\varphi^{-1} \cap \Delta^+\varphi^{-1} = H\varphi^{-1} \cap (\Delta^+ \cap \Delta^{\hat{0}}) \\ &= (H\varphi^{-1} \cap \Delta^{\hat{0}}) \cap \Delta^+ = H\varphi^{-1} \cap \Delta^+ = (H\varphi^{-1})^+,\end{aligned}$$

$(\mathcal{H}^+)^{\varphi^{-1}}$ and $(\mathcal{H}^{\varphi^{-1}})^+$ have the same hypermap subgroup, and hence are isomorphic. □

As a by-product of Propositions 1.7.4 and 1.6.3, we get:

Corollary 1.7.5. *For every hypermap $\mathcal{H}$,* $\mathrm{Walsh}(\mathcal{H}^+) \cong \mathrm{Walsh}(\mathcal{H})^+$ *and* $\mathrm{Pin}(\mathcal{H}^+) \cong \mathrm{Pin}(\mathcal{H})^+$.

1.8 The closure cover and the covering core

Given a hypermap subgroup H of a hypermap $\mathcal{H}$, the core of H in Δ, H_Δ, is the largest normal subgroup of Δ contained in H, and the closure of H in Δ, H^Δ, is the smallest normal subgroup of Δ containing H. When H has finite index in Δ, H_Δ and H^Δ also have finite index in Δ, by Remark A.1.4, respectively. These 2 normal subgroups of Δ give rise to 2 regular hypermaps, the *covering core* of $\mathcal{H}$, $\mathcal{H}_\Delta$, with hypermap subgroup H_Δ, and the *closure cover* of $\mathcal{H}$, $\mathcal{H}^\Delta$, with hypermap subgroup H^Δ. The covering core of $\mathcal{H}$, $\mathcal{H}_\Delta$, is the smallest regular hypermap covering $\mathcal{H}$, and the closure cover of $\mathcal{H}$, $\mathcal{H}^\Delta$, is the largest regular hypermap covered by $\mathcal{H}$. When H is regular, $H_\Delta = H = H^\Delta$ and $\mathcal{H}_\Delta = \mathcal{H} = \mathcal{H}^\Delta$.

The next result is straightforward.

Lemma 1.8.1. *Let Θ be a normal subgroup of Δ and $\mathcal{H}$ a hypermap. Then:*

1. *$\mathcal{H}$ is Θ-conservative if and only if $\mathcal{H}^\Delta$ is Θ-conservative;*
2. *if $\mathcal{H}$ is Θ-conservative, then $\mathcal{H}_\Delta$ is Θ-conservative;*
3. *$\mathcal{H}$ is Θ-regular if and only if $\mathcal{H}^\Delta$ is Θ-regular;*
4. *if $\mathcal{H}$ is Θ-regular, then $\mathcal{H}_\Delta$ is Θ-regular.*

The converses of 2. and 4. may not be true (see Chapter 3 for counter-examples).

Remark 1.8.2. If $\mathcal{H} = (\Omega_\mathcal{H}, h_0, h_1, h_2)$ and $\mathcal{G} = (\Omega_\mathcal{G}, g_0, g_1, g_2)$ are hypermaps such that $\mathcal{H}$ covers $\mathcal{G}$ and $\mathcal{G}$ has no boundary, then $\mathcal{H}$ has no boundary either. Indeed, if $\psi : \mathcal{H} \to \mathcal{G}$ is a covering and g_i is fixed-point free, then h_i is also fixed-point free. When $\mathcal{H}$ is orientable, $\mathcal{H}$ and $\mathcal{H}^\Delta$ cover $\mathcal{T}_{\Delta^+} = \mathcal{D}_1$ and hence $\mathcal{H}$ and $\mathcal{H}^\Delta$ have no boundary. However, when $\mathcal{H}$ is non-orientable and without boundary, $\mathcal{H}^\Delta$ may have boundary. See Section 3.3 for examples.

When $\mathcal{H}$ is an orientable hypermap, $\mathcal{H}^{\Delta}$ and $\mathcal{H}_{\Delta}$ are also orientable. If $\mathcal{H}$ is non-orientable, then $\mathcal{H}^{\Delta}$ is also non-orientable, however $\mathcal{H}_{\Delta}$ may be orientable. In what follows we determine conditions for seeing if the covering core of a non-orientable hypermap is orientable or not.

Lemma 1.8.3. *Let $\mathcal{H}$ be a hypermap. Then $(\mathcal{H}^+)_{\Delta} \cong (\mathcal{H}_{\Delta})^+$ and $(\mathcal{H}^+)^{\Delta} \cong (\mathcal{H}^{\Delta})^+$.*

Theorem 1.8.4. *Let $\mathcal{H}$ be a non-orientable hypermap. Then $|\Omega_{\mathcal{H}_{\Delta}}| \leq |\Omega_{(\mathcal{H}^+)_{\Delta}}|$ and $\mathcal{H}_{\Delta}$ is orientable if and only if $|\Omega_{\mathcal{H}_{\Delta}}| = |\Omega_{(\mathcal{H}^+)_{\Delta}}|$.*

Proof. Since $\mathcal{H}^+ \to \mathcal{H}$, $(\mathcal{H}^+)_{\Delta} \to \mathcal{H}_{\Delta}$ and so $|\Omega_{\mathcal{H}_{\Delta}}| \leq |\Omega_{(\mathcal{H}^+)_{\Delta}}|$. Then $|\Omega_{\mathcal{H}_{\Delta}}| = |\Omega_{(\mathcal{H}^+)_{\Delta}}|$ if and only if $\mathcal{H}_{\Delta} \cong (\mathcal{H}^+)_{\Delta} \cong (\mathcal{H}_{\Delta})^+$, that is, if and only if $\mathcal{H}_{\Delta}$ is orientable. □

The next result relates the bipartite-type of a bipartite-uniform hypermap with the type of its closure cover and the type of its covering core.

Proposition 1.8.5. *Let $\mathcal{B}$ be a bipartite-uniform hypermap of type $(l_1, l_2; m; n)$.*

1. *If $\mathcal{B}^{\Delta}$ has no boundary, and has type (p, q, r), then $p \mid \gcd(l_1, l_2)$, $q \mid m$ and $r \mid n$.*

2. *$\mathcal{B}_{\Delta}$ has type $(\operatorname{lcm}(l_1, l_2), m, n)$.*

Proof. 1. Follows immediately from Lemma 1.1.1.
2. Since $\mathcal{B}$ is bipartite-uniform and $\operatorname{Mon}(\mathcal{B}) = \Delta/B_{\Delta} = \operatorname{Mon}(\mathcal{B}_{\Delta})$, $B_{\Delta}R_1R_2$ can be written as a product of disjoint cycles of length l_1 and l_2, and hence $B_{\Delta}R_1R_2$ has order $\operatorname{lcm}(l_1, l_2)$. Obviously, $B_{\Delta}R_2R_0$ and $B_{\Delta}R_0R_1$ have orders m and n, respectively. Therefore $\mathcal{B}_{\Delta}$ has type $(\operatorname{lcm}(l_1, l_2), m, n)$. □

1.9 Chirality groups and chirality indices

The definition of the chirality group and chirality index of an orientably-regular hypermap and it basic properties are due to Breda, Jones, Nedela and Škoviera [6]. The chirality group and the chirality index of a hypermap $\mathcal{H}$ can be regarded as algebraic and numerical measures of how far $\mathcal{H}$ deviates from being regular. However, in this thesis we use these concepts in a more general sense.

Let H be the hypermap subgroup of a hypermap $\mathcal{H}$. Because H_{Δ} is always a normal subgroup of H, we have a group

$$\Upsilon_{\Delta}(\mathcal{H}) = H/H_{\Delta} \tag{1.9}$$

called *upper chirality group* of $\mathcal{H}$. According to Lemma A.1.3, $\Upsilon_{\Delta}(\mathcal{H})$ is finite if $[\Delta : H]$ is finite. The size of $\Upsilon_{\Delta}(\mathcal{H})$, which can be computed dividing the number of flags of $\mathcal{H}_{\Delta}$ by the number of flags of $\mathcal{H}$, is called the *upper chirality index* and is denoted by $\iota_{\Delta}(\mathcal{H})$. Since the number of flags of $\mathcal{H}_{\Delta}$ is equal to the size of $\operatorname{Mon}(\mathcal{H})$, $\iota_{\Delta}(\mathcal{H}) = |\operatorname{Mon}(\mathcal{H})|/|\Omega_{\mathcal{H}}|$. However, H may not be normal in H^{Δ}. According to Lemma A.1.5, H is normal in H^{Δ} if and only if $\mathcal{H}$ is restrictedly-regular. The *lower chirality index*, denoted by $\iota^{\Delta}(\mathcal{H})$, is the index $[H^{\Delta} : H]$, which is finite whenever $[\Delta : H]$ is finite. We can compute $\iota^{\Delta}(\mathcal{H})$ dividing the number of flags of $\mathcal{H}$ by the number of flags of $\mathcal{H}^{\Delta}$. When H is a normal subgroup of H^{Δ}, we have another group, called the *lower chirality group*

$$\Upsilon^{\Delta}(\mathcal{H}) = H^{\Delta}/H. \tag{1.10}$$

Naturally, each of these groups is trivial if and only if $\mathcal{H}$ is regular.

If $\mathcal{H}$ is Θ-regular for some $\Theta \lhd_2 \Delta$, and $g \in \Delta \setminus \Theta$, then $H^\Delta = HH^g$, $H_\Delta = H \cap H^g$ (see Lemma A.1.7), H is a normal subgroup of H^Δ, and

$$\Upsilon^\Delta(\mathcal{H}) = H^\Delta/H = HH^g/H \cong H^g/(H \cap H^g) \cong H/H_\Delta = \Upsilon_\Delta(\mathcal{H}). \tag{1.11}$$

In this case, and whenever the upper and lower chirality groups are isomorphic we denote by $\Upsilon(\mathcal{H})$ the common group $\Upsilon^\Delta(\mathcal{H}) \cong \Upsilon_\Delta(\mathcal{H})$, called the *chirality group* of $\mathcal{H}$, and by $\iota(\mathcal{H})$ the common value $\iota^\Delta(\mathcal{H}) = \iota_\Delta(\mathcal{H})$, called the *chirality index* of $\mathcal{H}$.

If it is clear from the context, we write Υ and ι instead of $\Upsilon(\mathcal{H})$ and $\iota(\mathcal{H})$, for short.

Remark 1.9.1. When $\mathcal{H}$ is a Θ-regular hypermap, $H^\Delta \lhd \Theta$ and hence $\Upsilon^\Delta(\mathcal{H}) = H^\Delta/H$ is a normal subgroup of $\Theta/H \cong \mathrm{Aut}^\Theta(\mathcal{H})$, the group of Θ-conservative automorphisms of $\mathcal{H}$.

It follows from Corollary A.1.9 that $(H_\Delta)\overline{\sigma} = (H\overline{\sigma})_\Delta$ and $(H^\Delta)\overline{\sigma} = (H\overline{\sigma})^\Delta$, for all $\sigma \in S_{\{0,1,2\}}$. Consequently, the groups $\Upsilon_\Delta(\mathrm{D}_\sigma(\mathcal{H}))$ and $\Upsilon_\Delta(\mathcal{H})$ are isomorphic, as well as the groups $\Upsilon^\Delta(\mathrm{D}_\sigma(\mathcal{H}))$ and $\Upsilon^\Delta(\mathcal{H})$, when $\mathcal{H}$ is restrictedly-regular. In other words, dual hypermaps have the same upper and lower chirality groups.

The following result will be very useful to compute the chirality groups of the 2-restrictedly-regular hypermaps.

Lemma 1.9.2. *If Θ is a normal subgroup of Δ of index 2, $\mathcal{H}$ is a Θ-regular hypermap with hypermap subgroup $H = \langle T \rangle^\Theta$, and $g \in \{R_0, R_1, R_2\} \setminus \Theta$, then*

1. $H^\Delta = \langle T \cup T^g \rangle^\Theta$;
2. $\Upsilon^\Delta(\mathcal{H}) = \langle Ht^g \mid t \in T \rangle^{\Theta/H}$.

Proof. 1. Clearly, $H^\Delta \lhd \Theta$. Since $T \cup T^g \subseteq H^\Delta \lhd \Theta$, $\langle T \cup T^g \rangle^\Theta \subseteq H^\Delta$. On the other hand, $\langle T \cup T^g \rangle^g = \langle T^g \cup T^{g^2} \rangle = \langle T \cup T^g \rangle$, so $g \in \mathrm{N}_\Delta(\langle T \cup T^g \rangle) \subseteq \mathrm{N}_\Delta(\langle T \cup T^g \rangle^\Theta)$ (see Proposition A.1.6). Thus $\Delta = \langle g, \Theta \rangle \subseteq \mathrm{N}_\Delta(\langle T \cup T^g \rangle^\Theta)$, that is, $\langle T \cup T^g \rangle^\Theta \lhd \Delta$. Since $H = \langle T \rangle^\Theta \subseteq \langle T \cup T^g \rangle^\Theta \lhd \Delta$, $H^\Delta \subseteq \langle T \cup T^g \rangle^\Theta$.
2. Let $\pi : \Theta \to \Theta/H$ be the projection. Since

$$\begin{aligned} H^\Delta\pi &= \langle T \cup T^g \rangle^\Theta \pi \\ &= (\langle T \cup T^g \rangle \pi)^{\Theta/H} \\ &= (H\langle T \cup T^g \rangle/H)^{\Theta/H} \\ &= \langle Hs \mid s \in T \cup T^g \rangle^{\Theta/H} \\ &= \langle Ht^g \mid t \in T \rangle^{\Theta/H}, \end{aligned}$$

we get $\Upsilon^\Delta(\mathcal{H}) = H^\Delta/H = H^\Delta\pi = \langle Ht^g \mid t \in T \rangle^{\Theta/H}$. □

Computing the chirality group of $\mathrm{Walsh}(\mathcal{R})$ **and** $\mathrm{Pin}(\mathcal{R})$

In what follows we assume that φ is φ_W or φ_P, and that e is $e_W = R_2 {R_2}^{R_0}$ or $e_P = {R_1}^{R_0} {R_2}^{R_0}$, respectively. Then $\ker \varphi = \langle e \rangle^{\Delta^{\hat{0}}}$ (Lemma 1.6.2) and $\psi \circ \varphi = 1_\Delta$. We also assume that $\mathcal{R}$ is a regular hypermap with hypermap subgroup R, and T is a subset of Δ such that $R = \langle T \rangle^\Delta$.

Remark 1.9.3. Because $R\varphi^{-1} \subseteq \Delta^{\hat{0}}$, $(R\varphi^{-1})^{\Delta} \lhd \Delta^{\hat{0}}$ and

$$\Upsilon(\mathcal{R}^{\varphi^{-1}}) = (R\varphi^{-1})^{\Delta}/R\varphi^{-1} \lhd \Delta^{\hat{0}}/R\varphi^{-1} \cong \Delta/R = \mathrm{Aut}(\mathcal{R}), \tag{1.12}$$

that is, the chirality group of $\mathcal{R}^{\varphi^{-1}}$ is isomorphic to a normal subgroup of the automorphism group of $\mathcal{H}$. When $\mathcal{R}$ is orientable, $\mathcal{R}$ is orientably-regular and $R\varphi^{-1}$ is a normal subgroup of $\Delta^{+}\varphi^{-1} = \Delta^{+} \cap \Delta^{\hat{0}} = \Delta^{+0\hat{0}}$. It follows that the normal closure of $R\varphi^{-1}$, $(R\varphi^{-1})^{\Delta}$, is also a normal subgroup of $\Delta^{+0\hat{0}}$ and

$$\Upsilon(\mathcal{R}^{\varphi^{-1}}) = (R\varphi^{-1})^{\Delta}/R\varphi^{-1} \lhd \Delta^{+0\hat{0}}/R\varphi^{-1} \cong \Delta^{+}/R = \mathrm{Aut}^{+}(\mathcal{R}), \tag{1.13}$$

that is, the chirality group of $\mathcal{R}^{\varphi^{-1}}$ is isomorphic to a normal subgroup of the rotation group of $\mathcal{R}$.

Since φ is onto, by Proposition A.1.8, $\langle e, T\psi\rangle^{\Delta^{\hat{0}}}\varphi = \langle e, T\psi\rangle\varphi^{\Delta} = \langle e\varphi, T\psi\varphi\rangle^{\Delta} = \langle T\rangle^{\Delta}$. Thus, $\mathcal{R}^{\varphi^{-1}}$, has hypermap subgroup $R\varphi^{-1} = \langle e, T\psi\rangle^{\Delta^{\hat{0}}}\varphi\varphi^{-1} = \langle e, T\psi\rangle^{\Delta^{\hat{0}}} \ker\varphi = \langle e, T\psi\rangle^{\Delta^{\hat{0}}}$, because $\ker\varphi = \langle e\rangle^{\Delta^{\hat{0}}} \subseteq \langle e, T\psi\rangle^{\Delta^{\hat{0}}}$. This proves the following result.

Theorem 1.9.4 (Hypermap subgroups of Walsh($\mathcal{R}$) and Pin($\mathcal{R}$))**.** *Let $\mathcal{R}$ be a regular hypermap with hypermap subgroup $R = \langle T\rangle^{\Delta}$, for some subset T of Δ. Then $W := \langle R_2 {R_2}^{R_0}, T\psi\rangle^{\Delta^{\hat{0}}}$ and $P := \langle {R_1}^{R_0} {R_2}^{R_0}, T\psi\rangle^{\Delta^{\hat{0}}}$ are hypermap subgroups of $\mathcal{W} :=$ Walsh($\mathcal{R}$) and $\mathcal{P} :=$ Pin($\mathcal{R}$), respectively.*

As one can easily see, for each normal subgroup R of Δ we have a group isomorphism $\overline{\varphi} : \Delta^{\hat{0}}/R\varphi^{-1} \to \Delta/R$, $(R\varphi^{-1}g) \mapsto R(g\varphi)$. Indeed, $\overline{\varphi}$ is an homomorphism because φ is an homomorphism, $\overline{\varphi}$ is onto because φ is onto, and $\overline{\varphi}$ is one-to-one because for all $g \in \Delta^{\hat{0}}$, $(R\varphi^{-1})g \in \ker\overline{\varphi} \Leftrightarrow R(g\varphi) = R \Leftrightarrow g\varphi \in R \Leftrightarrow g \in R\varphi^{-1} \Leftrightarrow (R\varphi^{-1})g = R\varphi^{-1}$, that is, $\ker\overline{\varphi} = \{R\varphi^{-1}\}$. Then

$$\begin{aligned}
\Upsilon(\mathcal{R}^{\varphi^{-1}}) &\cong (\Upsilon(\mathcal{R}^{\varphi^{-1}}))\overline{\varphi} \\
&= ((R\varphi^{-1})^{\Delta}/R\varphi^{-1})\overline{\varphi} \\
&= (\langle R\varphi^{-1}t^{R_0} \mid t \in \{e\} \cup T\psi\rangle^{\Delta^{\hat{0}}/R\varphi^{-1}})\overline{\varphi} \\
&= (\langle R\varphi^{-1}t^{R_0} \mid t \in \{e\} \cup T\psi\rangle)\overline{\varphi}^{\Delta/R} \\
&= \langle (R\varphi^{-1}t^{R_0})\overline{\varphi} \mid t \in \{e\} \cup T\psi\rangle^{\Delta/R} \\
&= \langle R(t^{R_0}\varphi) \mid t \in \{e\} \cup T\psi\rangle^{\Delta/R}.
\end{aligned}$$

Let $\alpha_W, \alpha_P : \Delta^{\hat{0}} \to \Delta$ defined by $g\alpha_W = g\psi^{R_0}\varphi_W$ and $g\alpha_P = g\psi^{R_0}\varphi_P$. Then

$$R_0\alpha_{W} = R_1, \quad R_1\alpha_{W} = R_0, \quad R_2\alpha_{W} = R_2, \tag{1.14}$$

and

$$R_0\alpha_{P} = R_1, \quad R_1\alpha_{P} = R_0, \quad R_2\alpha_{P} = R_0. \tag{1.15}$$

Lemma 1.9.5. *If $\mathcal{R}$ is a regular hypermap with hypermap subgroup $R = \langle T\rangle^{\Delta}$, then*

$$\Upsilon(\mathrm{Walsh}(\mathcal{R})) \cong \langle Rs \mid s \in T\alpha_{W}\rangle^{\Delta/R}$$

and

$$\Upsilon(\mathrm{Pin}(\mathcal{R})) \cong \langle Rs \mid s \in \{R_1R_2\} \cup T\alpha_{P}\rangle^{\Delta/R}.$$

Proposition 1.9.6. *Let $\mathcal{R}$ be a regular hypermap of type (l,m,n) with hypermap subgroup R, $X = \{(R_1R_2)^l, (R_2R_0)^m, (R_0R_1)^n\}$, T a subset of Δ containing X and such that $R = \langle T\rangle^\Delta$, $S = T \setminus X$, $d_1 := \gcd(l,m)$ and $d_2 := \gcd(m,n)$. Then*

$$\Upsilon(\mathrm{Walsh}(\mathcal{R})) \cong \langle R(R_1R_2)^{d_1}, R(R_2R_0)^{d_1}, \{Rs\alpha_{_W} \mid s \in S\}\rangle^{\Delta/R}$$

and

$$\Upsilon(\mathrm{Pin}(\mathcal{R})) \cong \langle RR_1R_2, R(R_0R_1)^{d_2}, \{Rs\alpha_{_P} \mid s \in S\}\rangle^{\Delta/R},$$

Proof. We have

$$(R_1R_2)^l\alpha_{_W} = [(R_2R_0)^l]^{-1}, \quad (R_2R_0)^m\alpha_{_W} = [(R_1R_2)^m]^{-1}, \quad (R_0R_1)^n\alpha_{_W} = [(R_0R_1)^n]^{-1}$$

and

$$(R_1R_2)^l\alpha_{_P} = 1, \quad (R_2R_0)^m\alpha_{_P} = (R_0R_1)^m, \quad (R_0R_1)^n\alpha_{_P} = [(R_0R_1)^n]^{-1}.$$

To finish the proof, just note that if $Rg \in \Delta/R$ has order k, then $\langle Rg^p\rangle = \langle Rg^{\gcd(k,p)}\rangle$. $\square$

Using Proposition 1.9.6 together with Remark 1.9.3 we get:

Corollary 1.9.7. *Let $\mathcal{R}$ be a regular hypermap of type (l,m,n), $\mathcal{W} = \mathrm{Walsh}(\mathcal{R})$, $\mathcal{P} = \mathrm{Pin}(\mathcal{R})$, $d_1 := \gcd(l,m)$ and $d_2 := \gcd(m,n)$.*

1. *(a) If $d_1 = 1$ and $\mathcal{R}$ is orientable, then $\Upsilon(\mathcal{W}) \cong R\Delta^+/R = \Delta^+/R \cong \mathrm{Aut}^+(\mathcal{R})$ and $\mathcal{W}^\Delta$, having hypermap subgroup $\Delta^+{\varphi_{_W}}^{-1} = \Delta^{+0\hat{0}}$ is $\mathrm{Walsh}(\mathcal{S}_1) \cong \mathcal{S}_2$;*
   (b) If $d_1 = 1$ and $\mathcal{R}$ is non-orientable, then $\Upsilon(\mathcal{W}) \cong R\Delta^+/R = \Delta/R \cong \mathrm{Aut}(\mathcal{R})$, and $\mathcal{W}^\Delta$, having hypermap subgroup $\Delta{\varphi_{_W}}^{-1} = \Delta^{\hat{0}}$, is $\mathcal{T}_{\Delta^{\hat{0}}}$, a hypermap with boundary.
2. *(a) If $d_2 = 1$ and $\mathcal{R}$ is orientable, then $\Upsilon(\mathcal{P}) \cong R\Delta^+/R = \Delta^+/R \cong \mathrm{Aut}^+(\mathcal{R})$ and $\mathcal{P}^\Delta$, having hypermap subgroup $\Delta^+{\varphi_{_P}}^{-1} = \Delta^{+0\hat{0}}$, is $\mathrm{Pin}(\mathcal{S}_1) \cong \mathcal{S}_2$;*
   (b) If $d_2 = 1$ and $\mathcal{R}$ is non-orientable, then $\Upsilon(\mathcal{P}) \cong R\Delta^+/R = \Delta/R \cong \mathrm{Aut}(\mathcal{R})$ and $\mathcal{P}^\Delta$, having hypermap subgroup $\Delta{\varphi_{_P}}^{-1} = \Delta^{\hat{0}}$, is $\mathcal{T}_{\Delta^{\hat{0}}}$, a hypermap with boundary;
   (c) If $d_2 = 2$ and $\mathcal{R}$ is orientable and bipartite, then $\Upsilon(\mathcal{P}) = R\Delta^{+0\hat{0}}/R = \Delta^{+0\hat{0}}/R$ and $\mathcal{P}^\Delta$, having hypermap subgroup $\Delta^{+0\hat{0}}{\varphi_{_P}}^{-1}$, is $\mathrm{Pin}(\mathcal{S}_2) \cong \mathcal{S}_4$;
   (d) If $d_2 = 2$ and $\mathcal{R}$ is orientable but not bipartite, then $\Upsilon(\mathcal{P}) = R\Delta^{+0\hat{0}}/R = \Delta^+/R \cong \mathrm{Aut}^+(\mathcal{R})$ and $\mathcal{P}^\Delta$, having hypermap subgroup $\Delta^+{\varphi_{_P}}^{-1} = \Delta^{+0\hat{0}}$, is $\mathrm{Pin}(\mathcal{S}_1) \cong \mathcal{S}_2$.

1.10 Bipartite-regular hypermaps

For each $k \in \mathbb{N}$, let $\mathcal{M}_k$ be the regular map with hypermap subgroup

$$M_k := \langle (R_1R_2)^{2k}, (R_2R_0)^2, (R_0R_1)^{2k}, (R_2R_0)(R_0R_1)^{-k}\rangle^\Delta.$$

The map $\mathcal{M}_k$, denoted by $\{2k,2k\}_{1,0}$ in [33], is an orientable regular map with 1 face and $4k$ flags formed from a single $2k$-gon by identifying opposite edges orientably. The automorphism group of $\mathcal{M}_k$ is the dihedral group D_{2k}. Since $M_kR_1R_2 = M_kR_1R_0(R_0R_1)^{-k} = M_k(R_0R_1)^{-(k+1)}$, $M_kR_1R_2$ has order $2k/\gcd(2k,k+1) = 2k/\gcd(2,k+1)$, which is k if k is odd, and $2k$ if k is even. For this reason, the map $\mathcal{M}_k$ has type $(k,2,2k)$, if k is odd, or

$(2k, 2, 2k)$ if k is even. In other words, $\mathcal{M}_{2k+1}$ and $\mathcal{M}_{2k}$ have type $(2k+1, 2, 4k+2)$ and $(4k, 2, 4k)$, respectively. In addition, $\mathcal{M}_{2k+1}$ has 2 vertices, $2k+1$ edges, 1 face, euler characteristic $\chi = 2 + (2k+1) + 1 - 2(2k+1) = 2 - 2k$ (see Lemma 1.4.1) and genus $g = k$; $\mathcal{M}_{2k}$ has 1 vertices, $2k$ edges, 1 face, euler characteristic $\chi = 1 + 2k + 1 - 4k = 2 - 2k$ (see Lemma 1.4.1) and genus $g = k$. Hence, on each orientable surface of genus g there are, at least, two regular maps: $\mathcal{M}_{2g}$ and $\mathcal{M}_{2g+1}$.

The bipartite-regular hypermaps $\mathrm{Pin}(\mathcal{M}_{2k+1})$, $\mathrm{Pin}(\mathcal{M}_{2k})$, $\mathrm{Walsh}(\mathcal{M}_{2k+1})$ and $\mathrm{Walsh}(\mathcal{M}_{2k})$ have bipartite-types $(1, 2k+1; 4; 8k+4)$, $(1, 4k; 4; 8k)$, $(2, 2k+1; 2; 8k+4)$ and $(2, 4k; 2; 8k)$, respectively. The hypermap $\mathrm{Pin}(\mathcal{M}_1) \cong \mathcal{S}_4$ is regular; all others, being non-uniform, are bipartite-chiral. Because of this, on each orientable surface we can find bipartite-chiral and hence bipartite-regular hypermaps. Using Proposition 1.9.6, one can see that

$$\Upsilon(\mathrm{Walsh}(\mathcal{M}_k)) \cong \mathrm{Aut}(\mathcal{M}_k) \cong C_{2k},$$
$$\Upsilon(\mathrm{Pin}(\mathcal{M}_{2k})) \cong \mathrm{Aut}^+(\mathcal{M}_{2k}) \cong C_{4k}$$

and

$$\Upsilon(\mathrm{Pin}(\mathcal{M}_{2k+1})) \cong \mathrm{Aut}^{+0\hat{0}}(\mathcal{M}_{2k}) \cong C_{2k+1}.$$

We cannot ensure the existence of bipartite-regular hypermaps on each non-orientable surface using the Walsh and Pin constructions because $\mathrm{Walsh}(\mathcal{H})$, $\mathrm{Pin}(\mathcal{H})$ and $\mathcal{H}$ have the same underlying surface and because there are non-orientable surfaces with no regular hypermaps (see [78]). For instance, there are no regular hypermaps on the non-orientable surfaces with negative characteristic 0, 1, 16, 22, 25, 37, and 46. However, the epimorphism $\varphi_E : \Delta^{\hat{0}} \to \Delta$ defined by $R_1\varphi_E = R_1$, $R_2\varphi_E = R_2$, ${R_1}^{R_0}\varphi_E = R_0$ and ${R_2}^{R_0}\varphi_E = R_1$ gives rise to a construction of bipartite hypermaps with the following properties:

- $\mathcal{H}^{{\varphi_E}^{-1}}$ is orientable if and only if $\mathcal{H}$ is orientable;
- $V(\mathcal{H}^{{\varphi_E}^{-1}}) = V(\mathcal{H}) + F(\mathcal{H})$,
  $E(\mathcal{H}^{{\varphi_E}^{-1}}) = V(\mathcal{H})$,
  $F(\mathcal{H}^{{\varphi_E}^{-1}}) = F(\mathcal{H})$;
- $\chi(\mathcal{H}^{{\varphi_E}^{-1}}) = 2(\chi(\mathcal{H}) - E(\mathcal{H}))$. Indeed

$$\begin{array}{rcl} \chi(\mathcal{H}^{{\varphi_E}^{-1}}) & = & V(\mathcal{H}^{{\varphi_E}^{-1}}) + E(\mathcal{H}^{{\varphi_E}^{-1}}) + F(\mathcal{H}^{{\varphi_E}^{-1}}) - |\Omega_{\mathcal{H}^{{\varphi_E}^{-1}}}|/2 \\ & = & 2(V(\mathcal{H}) + F(\mathcal{H})) - 2|\Omega_{\mathcal{H}}|/2 \\ & = & 2(\chi(\mathcal{H}) - E(\mathcal{H})). \end{array}$$

- $\mathcal{H}$ is uniform of type (l, m, n) if and only if $\mathcal{H}^{{\varphi_E}^{-1}}$ is bipartite-uniform of bipartite-type $(l, n; l; 2n)$.

The non-orientable regular hypermap $\mathcal{N}_k$ (denoted by $\mathcal{PP}_{2k}$ in Chapter 3) with hypermap subgroup $N_k := \langle (R_1R_2)^2, (R_2R_0)^2, (R_0R_1)^{2k}, (R_0R_1)^kR_2\rangle^{\Delta}$ is a hypermap on the projective plane of type $(2, 2, 2k)$ with $4k$ flags, k vertices, k edges and 1 face. The automorphism group of $\mathcal{N}_k$ is the dihedral group D_{2k}. Then, the hypermap ${\mathcal{N}_k}^{{\varphi_E}^{-1}}$ is a non-orientable hypermap on a surface with Euler characteristic $\chi({\mathcal{N}_k}^{{\varphi_E}^{-1}}) = 2(\chi(\mathcal{N}_k) - E(\mathcal{N}_k)) = 2(1-k)$. This shows that we can find a bipartite-chiral hypermap on each non-orientable surface with even Euler characteristic.

The existence of bipartite-regular hypermaps on every non-orientable surface with odd Euler characteristic remains an open problem.

Chapter 2

Hypermaps on the sphere

In this chapter we classify the 2-restrictedly-regular hypermaps on the sphere using the results obtained in Chapter 1. It is well-known that all uniform hypermaps on the sphere are regular and hence all 2-restrictedly-regular hypermaps on the sphere are bipartite-chiral.

The next section is included here for completeness.

2.1 Uniform hypermaps on the sphere

Let $\mathcal{U}$ be a uniform hypermap on the sphere of type (l, m, n). Using the Euler formula for uniform hypermaps (Corollary 1.4.2) together with Lemma 1.4.5 and Lemma 1.4.4, one can see that the type (l, m, n) of a uniform hypermap $\mathcal{U}$ on the sphere is, up to duality, $(1, k, k)$, $(2, 2, k)$, $(2, 3, 3)$, $(2, 3, 4)$ or $(2, 3, 5)$.

The following result is well-known.

Theorem 2.1.1 (Hypermap subgroups of the uniform hypermaps on the sphere)**.** *If $\mathcal{U}$ is a uniform hypermap on the sphere of type (l, m, n), then $\mathcal{U}$ has hypermap subgroup $N(l, m, n) = \langle (R_1R_2)^l, (R_2R_0)^m, (R_0R_1)^n \rangle^\Delta$.*

Proof. Let U be a hypermap subgroup of $\mathcal{U}$ and $N := \langle (R_1R_2)^l, (R_2R_0)^m, (R_0R_1)^n \rangle^\Delta$. Then $N \subseteq U$. By inspection one can see that $[\Delta : N] = [\Delta : U]$ and hence $N = U$. □

Corollary 2.1.2. *Uniform hypermaps on the sphere of the same type are isomorphic.*

Thus, up to duality, the unique uniform hypermaps on the sphere are the 2 infinite families $\mathcal{D}_k$ and $\mathcal{P}_k$, the tetrahedron $\mathcal{T}$, the cube (or hexahedron) $\mathcal{C}$ and the dodecahedron $\mathcal{D}$.

Corollary 2.1.3 (Conservativeness of uniform hypermaps on the sphere)**.** *Let $\Theta \triangleleft_2 \Delta$. Then:*

1. *(a) $\mathcal{D}_{2k-1}$ is Θ-conservative if and only if $\Theta = \Delta^+$;*

   (b) $\mathcal{D}_{2k}$ is Θ-conservative if and only if $\Theta \in \{\Delta^+, \Delta^2, \Delta^{\hat{2}}\}$;

2. *(a) $\mathcal{P}_{2k-1}$ is Θ-conservative if and only if $\Theta \in \{\Delta^+, \Delta^2, \Delta^{\hat{2}}\}$;*

   (b) $\mathcal{P}_{2k}$ is Θ-conservative;

3. *$\mathcal{T}$ is Θ-conservative if and only if $\Theta = \Delta^+$;*

4. *$\mathcal{C}$ is Θ-conservative if and only if $\Theta \in \{\Delta^+, \Delta^0, \Delta^{\hat{0}}\}$;*

5. $\mathcal{D}$ is Θ-conservative if and only if $\Theta = \Delta^+$.

Proof. Given $\Theta \lhd \Delta$, $\langle (R_1R_2)^l, (R_2R_0)^m, (R_0R_1)^n \rangle^\Delta$ is a subset of Θ if and only if $(R_1R_2)^l$, $(R_2R_0)^m$ and $(R_0R_1)^n$ belong to Θ. $\square$

The following well-known result is also an immediate corollary of Theorem 2.1.1.

Theorem 2.1.4. *All uniform hypermaps on the sphere are regular.*

Corollary 2.1.5. *If $\mathcal{U}$ is a uniform hypermap on the sphere, then $\mathcal{U}$ is Θ-regular if and only if $\mathcal{U}$ is Θ-conservative.*

Corollary 2.1.6. *There are no 2-restrictedly-regular uniform hypermaps on the sphere. In particular, there are no orientably-chiral or pseudo-orientably-chiral hypermaps on the sphere.*

Table 2.1 lists, up to duality, all possible values (l, m, n) for the type of a uniform hypermap $\mathcal{U}$ on the sphere. It also displays the numbers V of vertices, E of edges, F of faces and $|\Omega_\mathcal{U}|$ of flags of $\mathcal{U}$, as well as its symmetry and rotation groups. Finally, in the last column, we give the unique uniform hypermap on the sphere of type (l, m, n).

#	l	m	n	V	E	F	$\lvert\Omega_\mathcal{U}\rvert$	Aut($\mathcal{U}$)	Aut$^+$($\mathcal{U}$)	$\mathcal{U}$
1	1	k	k	k	1	1	$2k$	D_k	C_k	$\mathcal{S}_k$
2	2	2	k	k	k	2	$4k$	$D_k \times C_2$	D_k	$\mathcal{P}_k$
3	3	2	3	4	6	4	24	S_4	A_4	$\mathcal{T}$
4	3	2	4	8	12	6	48	$S_4 \times C_2$	S_4	$\mathcal{C}$
5	3	2	5	20	30	12	120	$A_5 \times C_2$	A_5	$\mathcal{D}$

Table 2.1: The uniform hypermaps on the sphere, up to duality.

Because the sphere is an orientable surface, every hypermap on the sphere is orientable, that is, Δ^+-conservative. Having in mind that $\Delta^{+k\hat{k}} = \Delta^+ \cap \Delta^k = \Delta^+ \cap \Delta^{\hat{k}}$, Corollary 2.1.5 implies that a uniform hypermap $\mathcal{U}$ on the sphere is Δ^k-regular if and only if $\mathcal{U}$ is $\Delta^{\hat{k}}$-regular. In Table 2.2, we display, up to duality, the Θ-regularity of the uniform hypermaps on the sphere, for each $\Theta \lhd_2 \Delta$.

#	$\mathcal{U}$	Δ^+-regular?	Δ^0-,$\Delta^{\hat{0}}$-regular?	Δ^1-,$\Delta^{\hat{1}}$-regular?	Δ^2-,$\Delta^{\hat{2}}$-regular?
1	$\mathcal{S}_k$	yes	yes iff $2 \mid k$	no	no
2	$\mathcal{P}_k$	yes	yes iff $2 \mid k$	yes iff $2 \mid k$	yes
3	$\mathcal{T}$	yes	no	no	no
4	$\mathcal{C}$	yes	yes	no	no
5	$\mathcal{D}$	yes	no	no	no

Table 2.2: Θ-regularity of the uniform hypermaps on the sphere

2.2 Bipartite-uniform hypermaps on the sphere

Let $\mathcal{B}$ be a bipartite-uniform hypermap on the sphere of bipartite-type $(l_1, l_2; m; n)$. We may assume, without loss of generality, that $l_1 \leq l_2$ and $m \leq n$. Then, by Lemma 1.3.6, m and

n are even. Replacing $\chi_{\mathcal{B}} = 2 > 0$ in the Euler formula for bipartite-uniform hypermaps (Corollary 1.4.3), one has that $(a, b, c, d) = (l_1, l_2, m/2, n/2)$ is a solution of the inequation $1/a + 1/b + 1/c + 1/d > 2$. According to Lemma 1.4.7, $l_1 = 1$ or $m/2 = 1$. Using Theorems 1.6.5 and 1.6.9, we get the following result.

Theorem 2.2.1. *If $\mathcal{B}$ is a bipartite-uniform hypermap on the sphere, then $\mathcal{B} \cong \mathrm{Walsh}(\mathcal{U})$ or $\mathcal{B} \cong \mathrm{Pin}(\mathcal{U})$ for some uniform hypermap $\mathcal{U}$ on the sphere, unique up to isomorphism. Moreover, as $\mathcal{B}$ is bipartite-regular if and only if $\mathcal{U}$ is regular, and on the sphere all uniform hypermaps are regular, then all bipartite-uniform hypermaps on the sphere are bipartite-regular.*

The solution $(a, b, c, d) = (1, 1, j, k)$ of $1/a+1/b+1/c+1/d > 2$ gives rise to the bipartite-types $(1, 1; 2j; 2k)$, $(1, j; 2; 2k)$ and $(j, k; 2; 2)$. By Theorems 1.6.4 and 1.6.8, a bipartite-uniform hypermap $\mathcal{B}$ with one of these bipartite-types is isomorphic to $\mathrm{Walsh}(\mathcal{U})$ or $\mathrm{Pin}(\mathcal{U})$, where $\mathcal{U}$ is a uniform hypermap on the sphere of type, up to duality, $(1, j, k)$. By Lemma 1.4.4, $j = k$.

Using Theorems 1.6.5 and 1.6.9 together with Corollary 2.1.2 and Lemma 1.6.1, we get:

Corollary 2.2.2. *Bipartite-uniform hypermaps on the sphere of the same bipartite-type are isomorphic.*

Table 2.3 lists, up to duality, all possible values $(l_1, l_2; m; n)$ for the bipartite-type of a bipartite-uniform hypermap $\mathcal{B}$ on the sphere, which are given by Lemma 1.4.7. We also display the numbers V_1 and V_2 of vertices in each $\Delta^{\hat{0}}$-orbit, E of edges, F of faces and $|\Omega_{\mathcal{B}}|$ of flags. In the last column of Table 2.3, we give the unique bipartite-uniform hypermap with such bipartite-type. We remark that the bipartite-uniform map of bipartite-type $(1, n; 2; 2n)$ can be obtained from $\mathrm{D}_{(12)}(\mathcal{D}_n)$ either via a Walsh construction or via a Pin construction. Indeed $\mathrm{Walsh}(\mathrm{D}_{(12)}(\mathcal{D}_n)) \cong \mathrm{Pin}(\mathrm{D}_{(12)}(\mathcal{D}_n))$. Notice that the hypermaps on lines 20, 21 and 22 are the 2-skeletons of the cube, the rhombic triacontahedron and the rhombic dodecahedron. These last two are Catalan solids or Archimedean duals (see §2.7 of [31]); their dual polyhedrons are the icosidodecahedron and the cuboctahedron, respectively.

As a by-product of Theorems 2.1.4 and 2.2.1 we have:

Theorem 2.2.3. *For every $\Theta \lhd \Delta$ with $[\Delta : \Theta] \leq 2$, Θ-uniformity on the sphere implies Θ-regularity.*

The existence of a normal subgroup Θ of Δ for which Θ-uniformity on the sphere does not imply Θ-regularity remains an open problem.

2.3 Chirality groups and chirality indices of the 2-restrictedly-regular hypermaps on the sphere

As we have mentioned before, every orientably-regular or pseudo-orientably-regular hypermap on the sphere is regular, so their chirality groups are trivial and their chirality indices are 1. In addition, all 2-restrictedly-regular hypermaps on the sphere are bipartite-chiral.

In this section we compute the chirality groups and the chirality indices of the bipartite-regular hypermaps on the sphere using the notations of Proposition 1.9.6.

#	l_1	l_2	m	n	V_1	V_2	E	F	$\|\Omega_\mathcal{B}\|$	$\mathcal{B}$
1	1	1	$2k$	$2k$	k	k	1	1	$4k$	$\mathrm{Pin}(\mathrm{D}_{(02)}(\mathcal{D}_k))$
2	1	2	4	$2k$	$2k$	k	k	2	$8k$	$\mathrm{Pin}(\mathcal{P}_k)$
3	1	2	6	6	12	6	4	4	48	$\mathrm{Pin}(\mathrm{D}_{(01)}(\mathcal{T}))$
4	1	2	6	8	24	12	8	6	96	$\mathrm{Pin}(\mathrm{D}_{(01)}(\mathcal{C}))$
5	1	2	6	10	60	30	20	12	240	$\mathrm{Pin}(\mathrm{D}_{(01)}(\mathcal{D}))$
6	1	3	4	6	12	4	6	4	48	$\mathrm{Pin}(\mathcal{T})$
7	1	3	4	8	24	8	12	6	96	$\mathrm{Pin}(\mathcal{C})$
8	1	3	4	10	60	20	30	12	240	$\mathrm{Pin}(\mathcal{D})$
9	1	4	4	6	24	6	12	8	96	$\mathrm{Pin}(\mathrm{D}_{(02)}(\mathcal{C}))$
10	1	5	4	6	60	12	30	20	240	$\mathrm{Pin}(\mathrm{D}_{(02)}(\mathcal{D}))$
11	1	k	2	$2k$	k	1	k	1	$4k$	$\mathrm{Pin}(\mathrm{D}_{(12)}(\mathcal{D}_k)) \cong$ $\cong \mathrm{Walsh}(\mathrm{D}_{(12)}(\mathcal{D}_k))$
12	1	k	4	4	$2k$	2	k	k	$8k$	$\mathrm{Pin}(\mathrm{D}_{(02)}(\mathcal{P}_k))$
13	2	2	2	$2k$	k	k	$2k$	2	$8k$	$\mathrm{Walsh}(\mathcal{P}_k)$
14	2	3	2	6	6	4	12	4	48	$\mathrm{Walsh}(\mathcal{T})$
15	2	3	2	8	12	8	24	6	96	$\mathrm{Walsh}(\mathcal{C})$
16	2	3	2	10	30	20	60	12	240	$\mathrm{Walsh}(\mathcal{D})$
17	2	4	2	6	12	6	24	8	96	$\mathrm{Walsh}(\mathrm{D}_{(02)}(\mathcal{C}))$
18	2	5	2	6	30	12	60	20	240	$\mathrm{Walsh}(\mathrm{D}_{(02)}(\mathcal{D}))$
19	2	k	2	4	k	2	$2k$	k	$8k$	$\mathrm{Walsh}(\mathrm{D}_{(02)}(\mathcal{P}_k))$
20	3	3	2	4	4	4	12	6	48	$\mathrm{Walsh}(\mathrm{D}_{(12)}(\mathcal{T}))$
21	3	4	2	4	8	6	24	12	96	$\mathrm{Walsh}(\mathrm{D}_{(12)}(\mathcal{C}))$
22	3	5	2	4	20	12	60	30	240	$\mathrm{Walsh}(\mathrm{D}_{(12)}(\mathcal{D}))$
23	k	k	2	2	1	1	k	k	$4k$	$\mathrm{Walsh}(\mathcal{D}_k)$

Table 2.3: The bipartite-regular hypermaps on the sphere.

Chirality groups and chirality indices of $\mathcal{B} = \mathrm{Walsh}(\mathcal{R})$

In what follows we assume that $\mathcal{R}$ is a regular hypermap on the sphere of type (l, m, n) and $\mathcal{B} = \mathrm{Walsh}(\mathcal{R})$. According to Proposition 1.9.6, $T = \{(R_1R_2)^l, (R_2R_0)^m, (R_0R_1)^n\}$ and $S = \varnothing$.

According to Table 2.3, up to duality, there are 12 types of bipartite-regular hypermaps on the sphere obtained from regular hypermaps using the Walsh construction.

When $l = m$, $\mathcal{B}$ is uniform and hence regular. After all, if $l = m$, then $d_1 = l = m$, and $\Upsilon(\mathcal{B}) \cong \langle R(R_1R_2)^{d_1}, R(R_2R_0)^{d_1}\rangle^{\Delta/R} = 1^{\Delta/R} = 1$. In addition, $\mathcal{B}^\Delta = \mathcal{B}$.

If $d_1 = 1$, then, according to Corollary 1.9.7, $\Upsilon(\mathcal{B}) = \Delta^+/R \cong \mathrm{Aut}^+(\mathcal{R})$ and $\mathcal{B}^\Delta$ is $\mathcal{S}_2$.

Table 2.4 lists the 12 types of bipartite-regular hypermaps on the sphere obtained from regular hypermaps using the Walsh construction. Of those cases, only 2 are non-uniform with $d_1 \neq 1$: cases 17 and 19 (k even). The chirality groups of these hypermaps are computed below. The last two columns of Table 2.4 display the chirality groups and chirality indices.

- Case **17**: $\mathcal{B} = \mathrm{Walsh}(\mathcal{R})$, $\mathcal{R} = \mathrm{D}_{(02)}(\mathcal{C})$ has type $(4, 2, 3)$ and $d_1 = 2$. Then $\Upsilon(\mathcal{B}) \cong \langle R(R_1R_2)^2, R(R_2R_0)^2\rangle^{\Delta/R} = \langle R(R_1R_2)^2, R((R_1R_2)^2)^{R_0}\rangle \cong V_4$ and $\iota = 4$; $\mathcal{B}^\Delta$ is $\mathcal{P}_6$: $\mathcal{R} \to \mathcal{P}_3$, $\mathcal{B} \to \mathrm{Walsh}(\mathcal{P}_3) \cong \mathcal{P}_6$, $\mathcal{P}_6$ is regular and $|\Omega_\mathcal{B}| = 96 = \iota|\Omega_{\mathcal{P}_6}|$.

- Case **19**: $\mathcal{B} = \text{Walsh}(\mathcal{R})$, $\mathcal{R} = \text{D}_{(02)}(\mathcal{P}_{2k})$ has type $(2k, 2, 2)$ and $d_1 = 2$. Then
$\Upsilon(\mathcal{B}) \cong \langle R(R_1R_2)^2, R(R_2R_0)^2\rangle^{\Delta/R} = \langle R(R_1R_2)^2\rangle \cong C_k$ and $\iota = k$;
$\mathcal{B}^\Delta$ is $\mathcal{P}_4$: $\mathcal{R} \to \mathcal{P}_2$, $\mathcal{B} \to \text{Walsh}(\mathcal{P}_2) \cong \mathcal{P}_4$, $\mathcal{P}_4$ is regular and $|\Omega_\mathcal{B}| = 16k = \iota|\Omega_{\mathcal{P}_4}|$.

#	$\mathcal{B} = \text{Walsh}(\mathcal{R})$	type of $\mathcal{R}$	$\text{Aut}(\mathcal{R})$	$\text{Aut}^+(\mathcal{R})$	d_1	Υ	ι
11	$\text{Walsh}(\text{D}_{(12)}(\mathcal{D}_k))$	$(k,1,k)$	D_k	C_k	1	C_k	k
13	$\text{Walsh}(\mathcal{P}_k)$	$(2,2,2k)$	$D_k \times C_2$	D_k	2	1	1
14	$\text{Walsh}(\mathcal{T})$	$(3,2,3)$	S_4	A_4	1	A_4	12
15	$\text{Walsh}(\mathcal{C})$	$(3,2,4)$	$S_4 \times C_2$	S_4	1	S_4	24
16	$\text{Walsh}(\mathcal{D})$	$(3,2,5)$	$A_5 \times C_2$	A_5	1	A_5	60
17	$\text{Walsh}(\text{D}_{(02)}(\mathcal{C}))$	$(4,2,3)$	$S_4 \times C_2$	S_4	2	V_4	4
18	$\text{Walsh}(\text{D}_{(02)}(\mathcal{D}))$	$(5,2,3)$	$A_5 \times C_2$	A_5	1	A_5	60
19	$\text{Walsh}(\text{D}_{(02)}(\mathcal{P}_{2k}))$	$(2k,2,2)$	$D_{2k} \times C_2$	D_{2k}	2	C_k	k
	$\text{Walsh}(\text{D}_{(02)}(\mathcal{P}_{2k-1}))$	$(2k-1,2,2)$	$D_{2k-1} \times C_2$	D_{2k-1}	1	D_{2k-1}	$4k-2$
20	$\text{Walsh}(\text{D}_{(12)}(\mathcal{T}))$	$(3,3,2)$	S_4	A_4	3	1	1
21	$\text{Walsh}(\text{D}_{(12)}(\mathcal{C}))$	$(3,4,2)$	$S_4 \times C_2$	S_4	1	S_4	24
22	$\text{Walsh}(\text{D}_{(12)}(\mathcal{D}))$	$(3,5,2)$	$A_5 \times C_2$	A_5	1	A_5	60
23	$\text{Walsh}(\mathcal{D}_k)$	$(k,k,1)$	D_k	C_k	k	C_k	k

Table 2.4: The bipartite-regular hypermaps obtained by the Walsh construction.

Chirality groups and chirality indices of $\mathcal{B} = \text{Pin}(\mathcal{R})$

Now we assume that $\mathcal{R}$ is a regular hypermap on the sphere of type (l,m,n) and $\mathcal{B} = \text{Pin}(\mathcal{R})$. As before, $T = \{(R_1R_2)^l, (R_2R_0)^m, (R_0R_1)^n\}$ and $S = \varnothing$.

According to Table 2.3, up to duality, there are 12 types of bipartite-regular hypermaps on the sphere obtained from regular hypermaps using the Pin construction.

In the first case, $\mathcal{B}$ is uniform and hence regular. In addition, $\mathcal{B}^\Delta = \mathcal{B}$.

If $d_2 = 1$, then, according to Corollary 1.9.7, $\Upsilon(\text{Pin}(\mathcal{R})) = \Delta^+/R \cong \text{Aut}^+(\mathcal{R})$ and $\mathcal{B}^\Delta$ is $\mathcal{S}_2$.

Table 2.5 lists the 12 types of bipartite-regular hypermaps on the sphere obtained from regular hypermaps using the Pin construction. Of those cases, only 4 are non-uniform with $d_2 \neq 1$: cases 2 (k even), 3, 7 and 12. The chirality groups of these hypermaps are computed below. The last two columns of Table 2.5 display the chirality groups and chirality indices.

- Case **2**: $\mathcal{B} = \text{Pin}(\mathcal{R})$, $\mathcal{R} = \mathcal{P}_{2k}$ has type $(2,2,2k)$ and $d_2 = 2$. Then
$\Upsilon(\mathcal{B}) \cong \langle RR_1R_2, R(R_0R_1)^2\rangle^{\Delta/R} (= R\Delta^{+0\hat{0}}/R = \Delta^{+0\hat{0}}/R) \cong D_k$ and $\iota = 2k$; $\mathcal{B}^\Delta$ is $\mathcal{S}_4$.

- Case **3**: $\mathcal{B} = \text{Pin}(\mathcal{R})$, $\mathcal{R} = \text{D}_{(01)}(\mathcal{T})$ has type $(2,3,3)$ and $d_2 = 3$. Then
$\Upsilon(\mathcal{B}) \cong \langle RR_1R_2, R(R_0R_1)^3\rangle^{\Delta/R} = \langle RR_1R_2, R(R_1R_2)^{R_0}\rangle \cong V_4$ and $\iota = 4$;
$\mathcal{B}^\Delta$ is $\mathcal{S}_6$: $\mathcal{R} \to \mathcal{S}_3$, $\mathcal{B} \to \text{Pin}(\mathcal{S}_3) \cong \mathcal{S}_6$, $\mathcal{S}_6$ is regular and $|\Omega_\mathcal{B}| = 48 = \iota|\Omega_{\mathcal{S}_6}|$.

- Case **7**: $\mathcal{B} = \text{Pin}(\mathcal{R})$, $\mathcal{R} = \mathcal{C}$ has type $(3,2,4)$ and $d_2 = 2$. Then
$\Upsilon(\mathcal{B}) \cong \langle RR_1R_2, R(R_0R_1)^2\rangle^{\Delta/R} (= \Delta^{+0\hat{0}}/R) \cong A_4$ and $\iota = 12$; $\mathcal{B}^\Delta$ is $\mathcal{S}_4$.

- Case **12**: $\mathcal{B} = \mathrm{Pin}(\mathcal{R})$, $\mathcal{R} = \mathrm{D}_{(02)}(\mathcal{P}_k)$ has type $(k, 2, 2)$ and $d_2 = 2$. Then $\Upsilon(\mathcal{B}) \cong \langle RR_1R_2, R(R_0R_1)^2\rangle^{\Delta/R}(= \Delta^{+0\hat{0}}/R) = \langle RR_1R_2\rangle \cong C_k$ and $\iota = k$; $\mathcal{B}^{\Delta}$ is $\mathcal{S}_4$.

#	$\mathcal{B} = \mathrm{Pin}(\mathcal{R})$	type of $\mathcal{R}$	$\mathrm{Aut}(\mathcal{R})$	$\mathrm{Aut}^+(\mathcal{R})$	d_2	Υ	ι
1	$\mathrm{Pin}(\mathrm{D}_{(02)}(\mathcal{D}_k))$	$(1, k, k)$	D_k	C_k	k	1	1
2	$\mathrm{Pin}(\mathcal{P}_{2k})$	$(2, 2, 2k)$	$D_{2k} \times C_2$	D_{2k}	2	D_k	$2k$
	$\mathrm{Pin}(\mathcal{P}_{2k-1})$	$(2, 2, 2k-1)$	$D_{2k-1} \times C_2$	D_{2k-1}	1	D_{2k-1}	$4k-2$
3	$\mathrm{Pin}(\mathrm{D}_{(01)}(\mathcal{T}))$	$(2, 3, 3)$	S_4	A_4	3	V_4	4
4	$\mathrm{Pin}(\mathrm{D}_{(01)}(\mathcal{C}))$	$(2, 3, 4)$	$S_4 \times C_2$	S_4	1	S_4	12
5	$\mathrm{Pin}(\mathrm{D}_{(01)}(\mathcal{D}))$	$(2, 3, 5)$	$A_5 \times C_2$	A_5	1	A_5	60
6	$\mathrm{Pin}(\mathcal{T})$	$(3, 2, 3)$	S_4	A_4	1	A_4	12
7	$\mathrm{Pin}(\mathcal{C})$	$(3, 2, 4)$	$S_4 \times C_2$	S_4	2	A_4	12
8	$\mathrm{Pin}(\mathcal{D})$	$(3, 2, 5)$	$A_5 \times C_2$	A_5	1	A_5	60
9	$\mathrm{Pin}(\mathrm{D}_{(02)}(\mathcal{C}))$	$(4, 2, 3)$	$S_4 \times C_2$	S_4	1	S_4	24
10	$\mathrm{Pin}(\mathrm{D}_{(02)}(\mathcal{D}))$	$(5, 2, 3)$	$A_5 \times C_2$	A_5	1	A_5	60
11	$\mathrm{Pin}(\mathrm{D}_{(12)}(\mathcal{D}_k))$	$(k, 1, k)$	D_k	C_k	1	C_k	k
12	$\mathrm{Pin}(\mathrm{D}_{(02)}(\mathcal{P}_k))$	$(k, 2, 2)$	$D_k \times C_2$	D_k	2	C_k	k

Table 2.5: The bipartite-regular hypermaps obtained by the Pin construction.

The closure covers and the covering cores

Table 2.6 lists the chirality groups and chirality indices of all bipartite-regular hypermaps on the sphere, as well as their closure covers. In Table 2.7 we display the type, number of flags and genus of the covering cores.

Note that if $\mathcal{B}$ is one of the bipartite-regular hypermaps listed in lines 1, 13, 20 and 23 of Table 2.3 (or Table 2.7), then $\mathcal{B}$ is regular and $\mathcal{B} = \mathcal{B}^{\Delta} = \mathcal{B}_{\Delta}$.

Looking at Table 2.7, one can see that there are two covering cores (not in the families) that are duals of maps with less than 100 edges. After all, if $\mathcal{B}$ is a map, $\mathcal{B}$ has less than 100 edges if and only if $|\Omega_{\mathcal{B}}| < 400$. The maps are $\mathrm{D}_{(01)}((\mathrm{Pin}(\mathrm{D}_{(01)}(\mathcal{T})))_{\Delta})$ with 48 edges and Petrie path of length 4, and $(\mathrm{Walsh}(\mathrm{D}_{(02)}(\mathcal{C})))_{\Delta}$ with 96 edges and Petrie path of length 6. In [70] we can find a list of all non-trivial regular with no more than 100 edges (the list is complete except perhaps at maps with 84 edges), these maps are $P(70)$ and $DP(190)$, pages 144 and 181 respectively. These can also be consulted in Wilson's Census of orientably-regular maps [69].

Note that the chirality index of a bipartite-regular hypermap can be any positive integer number. Moreover, cyclic groups and dihedral groups are chirality groups of bipartite-regular hypermaps.

#	$\mathcal{B}$	$\lvert\Omega_{\mathcal{B}}\rvert$	$\mathcal{B}^{\Delta}$	type of $\mathcal{B}^{\Delta}$	$\lvert\Omega_{\mathcal{B}^{\Delta}}\rvert$	Υ	ι
1	$\mathrm{Pin}(\mathrm{D}_{(02)}(\mathcal{D}_k))$	$4k$	$\mathcal{S}_{2k}$	$(1,2k,2k)$	$4k$	1	1
2	$\mathrm{Pin}(\mathcal{P}_{2k})$	$16k$	$\mathcal{S}_4$	$(1,4,4)$	8	D_k	$2k$
	$\mathrm{Pin}(\mathcal{P}_{2k-1})$	$16k-8$	$\mathcal{S}_2$	$(1,2,2)$	4	D_{2k-1}	$4k-2$
3	$\mathrm{Pin}(\mathrm{D}_{(01)}(\mathcal{T}))$	48	$\mathcal{S}_6$	$(1,6,6)$	12	V_4	4
4	$\mathrm{Pin}(\mathrm{D}_{(01)}(\mathcal{C}))$	96	$\mathcal{S}_2$	$(1,2,2)$	4	S_4	24
5	$\mathrm{Pin}(\mathrm{D}_{(01)}(\mathcal{D}))$	240	$\mathcal{S}_2$	$(1,2,2)$	4	A_5	60
6	$\mathrm{Pin}(\mathcal{T})$	48	$\mathcal{S}_2$	$(1,2,2)$	4	A_4	12
7	$\mathrm{Pin}(\mathcal{C})$	96	$\mathcal{S}_4$	$(1,4,4)$	8	A_4	12
8	$\mathrm{Pin}(\mathcal{D})$	240	$\mathcal{S}_2$	$(1,2,2)$	4	A_5	60
9	$\mathrm{Pin}(\mathrm{D}_{(02)}(\mathcal{C}))$	96	$\mathcal{S}_2$	$(1,2,2)$	4	S_4	24
10	$\mathrm{Pin}(\mathrm{D}_{(02)}(\mathcal{D}))$	240	$\mathcal{S}_2$	$(1,2,2)$	4	A_5	60
11	$\mathrm{Pin}(\mathrm{D}_{(12)}(\mathcal{D}_k))$	$4k$	$\mathcal{S}_2$	$(1,2,2)$	4	C_k	k
12	$\mathrm{Pin}(\mathrm{D}_{(02)}(\mathcal{P}_k))$	$8k$	$\mathcal{S}_4$	$(1,4,4)$	8	C_k	k
13	$\mathrm{Walsh}(\mathcal{P}_k)$	$8k$	$\mathcal{P}_{2k}$	$(2,2,2k)$	$8k$	1	1
14	$\mathrm{Walsh}(\mathcal{T})$	48	$\mathcal{S}_2$	$(1,2,2)$	4	A_4	12
15	$\mathrm{Walsh}(\mathcal{C})$	96	$\mathcal{S}_2$	$(1,2,2)$	4	S_4	24
16	$\mathrm{Walsh}(\mathcal{D})$	240	$\mathcal{S}_2$	$(1,2,2)$	4	A_5	60
17	$\mathrm{Walsh}(\mathrm{D}_{(02)}(\mathcal{C}))$	96	$\mathcal{P}_6$	$(2,2,6)$	24	V_4	4
18	$\mathrm{Walsh}(\mathrm{D}_{(02)}(\mathcal{D}))$	240	$\mathcal{S}_2$	$(1,2,2)$	4	A_5	60
19	$\mathrm{Walsh}(\mathrm{D}_{(02)}(\mathcal{P}_{2k}))$	$16k$	$\mathcal{P}_4$	$(2,2,4)$	16	C_k	k
	$\mathrm{Walsh}(\mathrm{D}_{(02)}(\mathcal{P}_{2k-1}))$	$16k-8$	$\mathcal{S}_2$	$(1,2,2)$	4	D_{2k-1}	$4k-2$
20	$\mathrm{Walsh}(\mathrm{D}_{(12)}(\mathcal{T}))$	48	$\mathcal{C}$	$(3,2,4)$	48	1	1
21	$\mathrm{Walsh}(\mathrm{D}_{(12)}(\mathcal{C}))$	96	$\mathcal{S}_2$	$(1,2,2)$	4	S_4	24
22	$\mathrm{Walsh}(\mathrm{D}_{(12)}(\mathcal{D}))$	240	$\mathcal{S}_2$	$(1,2,2)$	4	A_5	60
23	$\mathrm{Walsh}(\mathcal{D}_k)$	$4k$	$\mathrm{D}_{(02)}(\mathcal{P}_k)$	$(k,2,2)$	$4k$	1	1

Table 2.6: $\mathcal{B}$ and $\mathcal{B}^{\Delta}$

#	$\mathcal{B}$	type of $\mathcal{B}_\Delta$	$\lvert\Omega_{\mathcal{B}_\Delta}\rvert$	genus	Υ	ι
1	$\mathrm{Pin}(\mathrm{D}_{(02)}(\mathcal{D}_k))$	$(1, 2k, 2k)$	$4k$	0	1	1
2	$\mathrm{Pin}(\mathcal{P}_{2k})$ $\mathrm{Pin}(\mathcal{P}_{2k-1})$	$(2, 4, 4k)$ $(2, 4, 4k-2)$	$32k^2$ $16(2k-1)^2$	$2k^2 - 2k + 1$ $4(k-1)^2$	D_k D_{2k-1}	$2k$ $4k-2$
3	$\mathrm{Pin}(\mathrm{D}_{(01)}(\mathcal{T}))$	$(2, 6, 6)$	192	9	V_4	4
4	$\mathrm{Pin}(\mathrm{D}_{(01)}(\mathcal{C}))$	$(2, 6, 8)$	2304	121	S_4	24
5	$\mathrm{Pin}(\mathrm{D}_{(01)}(\mathcal{D}))$	$(2, 6, 10)$	14400	841	A_5	60
6	$\mathrm{Pin}(\mathcal{T})$	$(3, 4, 6)$	576	37	A_4	12
7	$\mathrm{Pin}(\mathcal{C})$	$(3, 4, 8)$	1152	85	A_4	12
8	$\mathrm{Pin}(\mathcal{D})$	$(3, 4, 10)$	14400	1141	A_5	60
9	$\mathrm{Pin}(\mathrm{D}_{(02)}(\mathcal{C}))$	$(4, 4, 6)$	2304	193	S_4	24
10	$\mathrm{Pin}(\mathrm{D}_{(02)}(\mathcal{D}))$	$(5, 4, 6)$	14400	1381	A_5	60
11	$\mathrm{Pin}(\mathrm{D}_{(12)}(\mathcal{D}_k))$	$(k, 2, 2k)$	$4k^2$	$\frac{(k-1)(k-2)}{2}$	C_k	k
12	$\mathrm{Pin}(\mathrm{D}_{(02)}(\mathcal{P}_k))$	$(k, 4, 4)$	$8k^2$	$(k-1)^2$	C_k	k
13	$\mathrm{Walsh}(\mathcal{P}_k)$	$(2, 2, 2k)$	$8k$	0	1	1
14	$\mathrm{Walsh}(\mathcal{T})$	$(6, 2, 6)$	576	25	A_4	12
15	$\mathrm{Walsh}(\mathcal{C})$	$(6, 2, 8)$	2304	121	S_4	24
16	$\mathrm{Walsh}(\mathcal{D})$	$(6, 2, 10)$	14400	841	A_5	60
17	$\mathrm{Walsh}(\mathrm{D}_{(02)}(\mathcal{C}))$	$(4, 2, 6)$	384	9	V_4	4
18	$\mathrm{Walsh}(\mathrm{D}_{(02)}(\mathcal{D}))$	$(10, 2, 6)$	14400	841	A_5	60
19	$\mathrm{Walsh}(\mathrm{D}_{(02)}(\mathcal{P}_{2k}))$ $\mathrm{Walsh}(\mathrm{D}_{(02)}(\mathcal{P}_{2k-1}))$	$(2k, 2, 4)$ $(4k-2, 2, 4)$	$16k^2$ $16(2k-1)^2$	$(k-1)^2$ $4(k-1)^2$	C_k D_{2k-1}	k $4k-2$
20	$\mathrm{Walsh}(\mathrm{D}_{(12)}(\mathcal{T}))$	$(3, 2, 4)$	48	0	1	1
21	$\mathrm{Walsh}(\mathrm{D}_{(12)}(\mathcal{C}))$	$(12, 2, 4)$	2304	97	S_4	24
22	$\mathrm{Walsh}(\mathrm{D}_{(12)}(\mathcal{D}))$	$(15, 2, 4)$	14400	661	A_5	60
23	$\mathrm{Walsh}(\mathcal{D}_k)$	$(k, 2, 2)$	$4k$	0	1	1

Table 2.7: $\mathcal{B}$ and $\mathcal{B}_\Delta$.

Chapter 3

Hypermaps on the projective plane

In this chapter we classify the 2-restrictedly-regular hypermaps on the projective plane. As on the sphere, we determine all uniform and bipartite-uniform hypermaps on the projective plane. First we derive the classification of the uniform hypermaps on the projective plane from the classification of uniform hypermaps on the sphere. All uniform hypermaps on the projective plane are regular maps and can be found in §8.6 of [33]. Next we see that, as on the sphere, all bipartite-uniform hypermaps on the projective plane are obtained from uniform hypermaps using a Walsh or a Pin construction, and hence are bipartite-regular.

The next section is included here for completeness.

3.1 Uniform hypermaps on the projective plane

Let $\mathcal{U}$ be a uniform hypermap on the projective plane. Then, by Theorem 1.7.1, the orientable double cover of $\mathcal{U}$, $\mathcal{U}^+ = \mathrm{Orient}(\mathcal{U})$ is a uniform map on the sphere with the same type of $\mathcal{U}$ and with even numbers of vertices, edges and faces. Because of this, $\mathcal{U}^+$ cannot be $\mathcal{D}_k$ (case 1 of Table 2.1) or $\mathcal{P}_{2k-1}$ (case 2 of Table 2.1). Thus, if $\mathcal{U}$ is a uniform hypermap on the projective plane, then, up to duality, $\mathcal{U}^+$ is $\mathcal{P}_{2k}$, $\mathcal{T}$, $\mathcal{C}$ or $\mathcal{D}$. Furthermore, $\mathrm{Aut}(\mathcal{U}^+)$ has an involutory orientation-reversing automorphism which is not a reflection.

Points on the sphere opposing along a diameter are called *antipodal points* or *antipodes*. If P and Q are antipodes, we also say that Q is the antipode of P and vice versa. The mapping Φ^{ap} that maps each point of the sphere to its antipode is an involutory orientation-reversing automorphism of the sphere. It is well-known that when $\mathcal{U}$ is $\mathcal{P}_{2k}$, $\mathcal{C}$ or $\mathcal{D}$, Φ^{ap} induces an involutory orientation-reversing automorphism $\varphi^{\mathrm{ap}}_{\mathcal{U}}$ of $\mathcal{U}$ which is not a reflection. If U is a hypermap subgroup of $\mathcal{U}$, then $\varphi^{\mathrm{ap}}_{\mathcal{U}}$ maps each flag Ug, with $g \in \Delta$, to $UA_{\mathcal{U}}g$, where $A_{\mathcal{P}_{2k}} := (R_0R_1)^kR_2$, for $k \in \mathbb{N}$, $A_{\mathcal{C}} := (R_0R_1R_2)^3$ and $A_{\mathcal{D}} := (R_0R_1R_2)^5$. These automorphisms give rise to the following uniform hypermaps on the projective plane formed by identifying antipodal points of the sphere: the *projective polygon* of order k, $\mathcal{PP}_{2k}$, of type $(2,2,2k)$, the *projective cube*, also known as the *Purse of Fortunatus* (cf. §21.34 of [32]) or *hemi-cube*, $\mathcal{PC}$, of type $(3,2,4)$, and the *projective dodecahedron*, $\mathcal{PD}$, of type $(3,2,5)$. Table 3.1 gives some information about these hypermaps, namely their numbers of vertices, edges, faces and flags, and their symmetry groups. We recall that by Proposition 1.7.1, the numbers of flags, vertices, edges and faces of $\mathcal{U}$ are half the numbers of flags, vertices, edges and faces of $\mathcal{U}^+$. The hypermaps $\mathcal{PO} = \mathrm{D}_{(02)}(\mathcal{PC})$ and $\mathcal{PI} = \mathrm{D}_{(02)}(\mathcal{PD})$ are called *projective octahedron* and *projective icosahedron*, respectively. Using the properties of Orient, it follows that $\mathcal{PO}^+ = \mathrm{D}_{(02)}(\mathcal{PC})^+ = \mathrm{D}_{(02)}(\mathcal{PC}^+) = \mathrm{D}_{(02)}(\mathcal{C}) = \mathcal{O}$ and $\mathcal{PI}^+ = \mathrm{D}_{(02)}(\mathcal{PD})^+ = \mathrm{D}_{(02)}(\mathcal{PD}^+) = \mathrm{D}_{(02)}(\mathcal{D}) = \mathcal{I}$. By inspection, one can see that $\mathcal{P}_{2k}$, $\mathcal{C}$ and $\mathcal{D}$ have no other involutory orientation-reversing automorphism which is not a reflection besides $\varphi^{\mathrm{ap}}_{\mathcal{P}_{2k}}$, $\varphi^{\mathrm{ap}}_{\mathcal{C}}$ and $\varphi^{\mathrm{ap}}_{\mathcal{D}}$, respectively, and that all involutory orientation-reversing automorphisms of $\mathcal{T}$ are reflections. Therefore, up to duality, the unique uniform hypermaps on the projective are the infinite family $\mathcal{PP}_{2k}$, $\mathcal{PC}$ and $\mathcal{PD}$. In [33], Coxeter and Moser denoted the uniform hypermaps $\mathcal{PC}$, $\mathcal{PO}$, $\mathcal{PD}$, $\mathcal{PI}$, $\mathcal{PP}_{2k}$ and $\mathrm{D}_{(02)}(\mathcal{PP}_{2k})$ by $\{4,3\}/2 = \{4,3\}_3$, $\{3,4\}/2 = \{3,4\}_3$, $\{5,3\}/2 = \{5,3\}_5$, $\{3,5\}/2 = \{3,5\}_5$, $\{2k,2\}/2$ and $\{2,2k\}/2$, respectively. The hypermap $\mathcal{PP}_{2k}$, of type $(2,2,2k)$, was denoted by $\mathcal{D}^{\diamond}_k$ in [15], and by δ_k in [73]. As remarked in [13], $\mathcal{PP}_2$ is a hypermap on the projective plane with hypermap subgroup Δ^{012} and with automorphism group $\mathrm{Aut}(\mathcal{PP}_2)$ is isomorphic to V_4.

Now let $\mathcal{U}$ be $\mathcal{PP}_{2k}$, $\mathcal{PC}$ or $\mathcal{PD}$, and let (l,m,n) be the type of $\mathcal{U}$. Furthermore, let $S = \{(R_1R_2)^l, (R_2R_0)^m, (R_0R_1)^n, A_{\mathcal{U}^+}\}$ and $S^+ = S \cap \Delta^+ = \{(R_1R_2)^l, (R_2R_0)^m, (R_0R_1)^n\}$. By Theorems 1.7.1 and 2.1.1, $\mathcal{U}$ has hypermap subgroup $U = \langle S^+\rangle^{\Delta}\{1, A_{\mathcal{U}^+}\}$, because $\langle S^+\rangle^{\Delta}$ is a hypermap subgroup of $\mathcal{U}^+$. Since $\langle S\rangle \subseteq U \subseteq \langle S\rangle^{\Delta}$, $U = \langle S\rangle^{\Delta}$ if and only if $U \lhd \Delta$, or equivalently, if and only if $U/U^+ \lhd \Delta/U^+$, because the projection $\pi : \Delta \to \Delta/U^+$ is an epimorphism and $U\pi = U/U^+$. In all cases $U^+A_{\mathcal{U}^+} \leftrightharpoons U^+R_0, U^+R_1, U^+R_2$, so $U/U^+ = \langle U^+A_{\mathcal{U}^+}\rangle \subseteq \mathrm{Z}(\mathrm{Aut}(\mathcal{U}^+))$. In addition, since $\mathrm{Aut}(\mathcal{U}^+)$ is $D_{2k} \times C_2$, $S_4 \times C_2$ or $A_5 \times C_2$, $\mathrm{Z}(\mathrm{Aut}(\mathcal{U}^+)) \cong C_2$ and hence $U/U^+ = \mathrm{Z}(\mathrm{Aut}(\mathcal{U}^+)) \lhd \mathrm{Aut}(\mathcal{U}^+) = \Delta/U^+$.

For simplicity, we extend the definition of $A_{\mathcal{U}}$ in the following way. If $\sigma \in \{0,1,2\}$ and $\mathcal{U}$ is $\mathcal{C}$ or $\mathcal{D}$, then $A_{\mathrm{D}_\sigma(\mathcal{U})} := A_{\mathcal{U}}\overline{\sigma}$. When $\mathcal{U} = \mathcal{PP}_{2k}$, $A_{\mathrm{D}_{(01)}(\mathcal{P}_{2k})} = A_{\mathcal{P}_{2k}} = (R_0R_1)^kR_2$, $A_{\mathrm{D}_{(012)}(\mathcal{P}_{2k})} = A_{\mathrm{D}_{(02)}(\mathcal{P}_{2k})} = (R_1R_2)^kR_0$, $A_{\mathrm{D}_{(021)}(\mathcal{P}_{2k})} = A_{\mathrm{D}_{(12)}(\mathcal{P}_{2k})} = (R_2R_0)^kR_1$.

Theorem 3.1.1 (Hypermap subgroups of the uniform hypermaps on the projective plane). *If $\mathcal{U}$ is a uniform hypermap on the projective plane of type (l,m,n), then $\mathcal{U}$ has hypermap subgroup $U = \langle (R_1R_2)^l, (R_2R_0)^m, (R_0R_1)^n, A_{\mathcal{U}^+}\rangle^{\Delta}$.*

Corollary 3.1.2. *Uniform hypermaps on the projective plane of the same type are isomorphic.*

Corollary 3.1.3 (Conservativeness of the uniform hypermaps on the projective plane). *Let $\Theta \lhd_2 \Delta$. Then:*

1. *(a) $\mathcal{PP}_{4k-2}$ is Θ-conservative if and only if Θ is Δ^0, Δ^1 or Δ^2;*

   (b) $\mathcal{PP}_{4k}$ is Θ-conservative if and only if Θ is $\Delta^{\hat{0}}$, $\Delta^{\hat{1}}$ or Δ^2;

2. *$\mathcal{PC}$ is Θ-conservative if and only if $\Theta = \Delta^0$;*

3. *$\mathcal{PD}$ is not Θ-conservative.*

Proof. Similar to the proof of Corollary 2.1.3. Given $\Theta \lhd \Delta$, $\langle (R_1R_2)^l, (R_2R_0)^m, (R_0R_1)^n, A_{\mathcal{U}}\rangle^{\Delta}$ is a subset of Θ if and only if $(R_1R_2)^l$, $(R_2R_0)^m$, $(R_0R_1)^n$ and $A_{\mathcal{U}}$ belong to Θ. □

As a by-product of Theorem 3.1.1 we get following result:

Theorem 3.1.4. *All uniform hypermaps on the projective plane are regular.*

Corollary 3.1.5. *If $\mathcal{U}$ is a uniform hypermap on the projective plane, then $\mathcal{U}$ is Θ-regular if and only if $\mathcal{U}$ is Θ-conservative.*

Corollary 3.1.6. *There are no 2-restrictedly-regular uniform hypermaps on the projective plane. In particular, there are no pseudo-orientably-chiral hypermaps on the projective plane.*

As on the sphere, all 2-restrictedly-regular hypermaps on the projective plane are bipartite-chiral.

Table 3.1 lists, up to duality, all values (l, m, n) for the type of a uniform hypermap $\mathcal{U}$ on the projective plane. It also displays the numbers V of vertices, E of edges, F of faces and $|\Omega_{\mathcal{U}}|$ of flags of $\mathcal{U}$, as well as its automorphism group. In the last column, we give the unique uniform hypermap on the sphere of such type. Notice that the automorphism groups of $\mathcal{PC}$ and $\mathcal{PD}$ are just Coxeter groups $G^{3,3,4} \cong S_4$ and $G^{3,5,5} \cong A_5$ (see [33]).

#	l	m	n	V	E	F	$\lvert\Omega_{\mathcal{U}}\rvert$	$\mathrm{Aut}(\mathcal{U})$	$\mathcal{U}$
2	2	2	$2k$	k	k	1	$4k$	D_{2k}	$\mathcal{PP}_{2k}$
4	3	2	4	4	6	3	24	S_4	$\mathcal{PC}$
5	3	2	5	10	15	6	60	A_5	$\mathcal{PD}$

Table 3.1: The uniform hypermaps on the projective plane, up to duality.

Because the projective plane is a non-orientable surface, no hypermap on the projective plane is orientable and hence orientably-regular. In addition, since $\Delta^k \cap \Delta^{\hat{k}} \subseteq \Delta^+$, a hypermap on the projective plane cannot be simultaneously Δ^k-conservative and $\Delta^{\hat{k}}$-conservative. In Table 3.2, we display, up to duality, the Θ-regularity of the uniform hypermaps on the projective plane, for each $\Theta \lhd_2 \Delta$. Note that the projective dodecahedron is not Θ-regular for any $\Theta \lhd_2 \Delta$.

#	$\mathcal{U}$	$\Delta^{\hat{0}}$-regular?	$\Delta^{\hat{1}}$-regular?	$\Delta^{\hat{2}}$-regular?	Δ^{0}-regular?	Δ^{1}-regular?	Δ^{2}-regular?
1	$\mathcal{P}_{2k}$	yes iff $2 \mid k$	yes iff $2 \mid k$	no	yes iff $2 \nmid k$	yes iff $2 \nmid k$	yes
2	$\mathcal{PC}$	no	no	no	yes	no	no
3	$\mathcal{PD}$	no	no	no	no	no	no

Table 3.2: Θ-regularity of the uniform hypermaps on the projective

3.2 Bipartite-uniform hypermaps on the projective plane

Let $\mathcal{B}$ be a bipartite-uniform hypermap on the projective plane of bipartite-type $(l_1, l_2; m; n)$. We may assume, without loss of generality, that $l_1 \leq l_2$ and $m \leq n$. Then, by Lemma 1.3.6, m and n are even. Since the orientable double cover of $\mathcal{B}$, $\mathcal{B}^+ = \mathrm{Orient}(\mathcal{B})$, is a bipartite-uniform on the sphere with the same bipartite-type of $\mathcal{B}$, $l_1 = 1$ or $m/2 = 1$ (see Section 2.2). Using Theorems 1.6.5 and 1.6.9, we get the following result.

Theorem 3.2.1. *If $\mathcal{B}$ is a bipartite-uniform hypermap on the projective plane, then $\mathcal{B} \cong$ $\mathrm{Walsh}(\mathcal{U})$ or $\mathcal{B} \cong \mathrm{Pin}(\mathcal{U})$ for some uniform hypermap $\mathcal{U}$ on the projective plane, unique up to isomorphism. Moreover, as $\mathcal{B}$ is bipartite-regular if and only if $\mathcal{U}$ is regular, and on the projective plane all uniform hypermaps are regular, then all bipartite-uniform hypermaps on the projective plane are bipartite-regular.*

Using Theorem 3.2.1 and Corollary 3.1.2 together with Theorems 1.6.6 and 1.6.10, we get:

Using Theorems 1.6.5 and 1.6.9 together with Corollary 3.1.2 and Lemma 1.6.1, we get:

Corollary 3.2.2. *Bipartite-uniform hypermaps on the projective plane of the same bipartite-type are isomorphic.*

Table 3.3 lists, up to duality, all possible values $(l_1, l_2; m; n)$ for the bipartite-type of a bipartite-uniform hypermap $\mathcal{B}$ on the projective plane. We also display the numbers V_1 and V_2 of vertices in each $\Delta^{\hat{0}}$-orbit, E of edges, F of faces and $|\Omega_{\mathcal{B}}|$ of flags. In the last column of Table 3.3, we give the unique bipartite-uniform hypermap with such bipartite-type.

#	l_1	l_2	m	n	V_1	V_2	E	F	$\|\Omega_{\mathcal{B}}\|$	$\mathcal{B}$
1	1	2	4	$4k$	$2k$	k	k	1	$8k$	Pin($\mathcal{PP}_{2k}$)
2	1	2	6	8	12	6	4	3	48	Pin(D$_{(01)}$($\mathcal{PC}$))
3	1	2	6	10	30	15	10	6	120	Pin(D$_{(01)}$($\mathcal{PD}$))
4	1	3	4	8	12	4	6	3	48	Pin($\mathcal{PC}$)
5	1	3	4	10	30	10	15	6	120	Pin($\mathcal{PD}$)
6	1	4	4	6	12	3	6	4	48	Pin(D$_{(02)}$($\mathcal{PC}$))
7	1	5	4	6	30	6	15	10	120	Pin(D$_{(02)}$($\mathcal{PD}$))
8	1	$2k$	4	4	$2k$	1	k	k	$8k$	Pin(D$_{(02)}$($\mathcal{PP}_{2k}$))
9	2	2	2	$4k$	k	k	$2k$	1	$8k$	Walsh($\mathcal{PP}_{2k}$)
10	2	3	2	8	6	4	12	3	48	Walsh($\mathcal{PC}$)
11	2	3	2	10	15	10	30	6	120	Walsh($\mathcal{PD}$)
12	2	4	2	6	6	3	12	4	48	Walsh(D$_{(02)}$($\mathcal{PC}$))
13	2	5	2	6	15	6	30	10	120	Walsh(D$_{(02)}$($\mathcal{PD}$))
14	2	$2k$	2	4	k	1	$2k$	k	$8k$	Walsh(D$_{(02)}$($\mathcal{PP}_{2k}$))
15	3	4	2	4	4	3	12	6	48	Walsh(D$_{(12)}$($\mathcal{PC}$))
16	3	5	2	4	10	6	30	15	120	Walsh(D$_{(12)}$($\mathcal{PD}$))

Table 3.3: The bipartite-regular hypermaps on the projective plane.

Because Walsh($\mathcal{H}^+$) $\cong$ Walsh($\mathcal{H}$)$^+$ and Pin($\mathcal{H}^+$) $\cong$ Pin($\mathcal{H}$)$^+$, if $\mathcal{B}$ is a bipartite-uniform hypermap on the projective plane obtained from the uniform hypermap $\mathcal{U}$ via the Walsh or Pin construction, then the orientable double cover of $\mathcal{B}$, $\mathcal{B}^+$, is obtained from the orientable double cover of $\mathcal{U}$, $\mathcal{U}^+$, via the same construction.

As a by-product of Theorems 3.1.4 and 3.2.1 we have:

Theorem 3.2.3. *For every $\Theta \lhd \Delta$ with $[\Delta : \Theta] \leq 2$, Θ-uniformity on the projective plane implies Θ-regularity.*

The existence of a normal subgroup Θ of Δ for which Θ-uniformity on the projective plane does not imply Θ-regularity remains an open problem.

3.3 Chirality groups and chirality indices of the 2-restrictedly-regular hypermaps on the projective plane

We have seen that on the projective plane there are no orientably-regular hypermaps, and that all pseudo-orientably-regular hypermaps on the projective plane are regular, so their chirality groups are trivial and their chirality indices are 1. Because of this, every 2-restrictedly-regular hypermap on the projective plane is bipartite-chiral.

In this section we compute the chirality groups and the chirality indices of the bipartite-regular hypermaps on the projective plane using the notations of Proposition 1.9.6.

Chirality groups and chirality indices of $\mathcal{B} = \mathrm{Walsh}(\mathcal{R})$

In what follows we assume that $\mathcal{R}$ is a regular hypermap on the projective plane of type (l, m, n) and $\mathcal{B} = \mathrm{Walsh}(\mathcal{R})$. According to Proposition 1.9.6, $T = \{(R_1R_2)^l, (R_2R_0)^m, (R_0R_1)^n, A_{\mathcal{R}^+}\}$ and $S = \{A_{\mathcal{R}^+}\}$.

According to Table 3.3, up to duality, there are 8 types of bipartite-regular hypermaps on the projective plane obtained from regular hypermaps using the Walsh construction.

When $l = m$, $\mathcal{B}$ is uniform and hence regular. In addition, $\mathcal{B}^\Delta = \mathcal{B}$.

If $d_1 = 1$, then, by Corollary 1.9.7, $\Upsilon(\mathcal{B}) = \Delta/R \cong \mathrm{Aut}(\mathcal{R})$ and $\mathcal{B}^\Delta$ is $\mathcal{T}_{\Delta^{\hat{0}}}$.

Table 3.4 lists the 8 types of bipartite-regular hypermaps on the projective plane obtained from regular hypermaps using the Walsh construction. Of those cases, only 2 are non-uniform with $d_1 \neq 1$: cases 12 and 14. The chirality groups of these hypermaps are computed below. In the last two columns of Table 3.4 we display the chirality groups and chirality indices.

- Case **12**: $\mathcal{B} = \mathrm{Walsh}(\mathcal{R})$, $\mathcal{R} = \mathrm{D}_{(02)}(\mathcal{PC}))$ has type $(4, 2, 3)$ and $d_1 = 2$. Then $A_{\mathcal{R}^+} = A_{\mathrm{D}_{(02)}(\mathcal{C})} = A_{\mathcal{C}}\overline{(02)} = (R_2R_1R_0)^3$, $(A_{\mathcal{R}^+})\alpha_W = (R_2R_0R_1)^3$, $\Upsilon(\mathcal{B}) \cong \langle R(R_1R_2)^2, R(R_2R_0)^2, R(R_2R_0R_1)^3\rangle^{\Delta/R} = \langle R(R_1R_2)^2, R((R_1R_2)^2)^{R_0}\rangle \cong V_4$ and $\iota = 4$; $\mathcal{B}^\Delta$ is $\mathcal{PP}_6$: $\mathcal{R} \to \mathcal{PP}_3$, $\mathcal{B} \to \mathrm{Walsh}(\mathcal{PP}_3) \cong \mathcal{PP}_6$, $\mathcal{PP}_6$ is regular and $|\Omega_{\mathcal{B}}| = 48 = \iota|\Omega_{\mathcal{PP}_6}|$.

- Case **14**: $\mathcal{B} = \mathrm{Walsh}(\mathcal{R})$, $\mathcal{R} = \mathrm{D}_{(02)}(\mathcal{PP}_{2k})$ has type $(2k, 2, 2)$ and $d_1 = 2$. Then $A_{\mathcal{R}^+} = A_{\mathrm{D}_{(02)}(\mathcal{P}_{2k})} = A_{\mathcal{P}_{2k}}\overline{(02)} = (R_2R_1)^kR_0$, $(A_{\mathcal{R}^+})\alpha_W = (R_2R_0)^kR_1$ and $\Upsilon(\mathcal{B}) \cong \langle R(R_1R_2)^2, R(R_2R_0)^2, R(R_2R_0)^kR_1\rangle^{\Delta/R} = \langle R(R_1R_2)^2, R(R_2R_0)^kR_1\rangle^{\Delta/R}$. If $2 \mid k$, then $\Upsilon(\mathcal{B}) \cong \langle R(R_1R_2)^2, RR_1\rangle^{\Delta/R} = \langle R(R_1R_2)^2, RR_1\rangle \cong D_k$ and $\iota = 2k$. In fact $\Upsilon(\mathcal{B}) \cong \langle R(R_1R_2)^2, RR_1\rangle = \Delta^{\hat{2}}/R$, since $R = R(R_2R_1)^k = RR_0 = RR_0{}^{R_2}$; $\mathcal{B}^\Delta$ is $\mathcal{T}_{\Delta^{\hat{0}1\hat{2}}}$. Else, if $2 \nmid k$, then $\Upsilon(\mathcal{B}) \cong \langle R(R_1R_2)^2, RR_2R_0R_1\rangle^{\Delta/R} = \langle RR_2R_0R_1\rangle \cong C_k$ and $\iota = k$. In fact $\Upsilon(\mathcal{B}) \cong \langle RR_2R_0R_1\rangle^{\Delta/R} = R\Delta^{012}/R = \Delta^{012}/R$; $\mathcal{B}^\Delta$ is $\mathcal{PP}_4$: $\mathcal{R} \to \mathcal{PP}_2$, $\mathcal{B} \to \mathrm{Walsh}(\mathcal{PP}_2) \cong \mathcal{PP}_4$, $\mathcal{PP}_4$ is regular and $|\Omega_{\mathcal{B}}| = 8k = \iota|\Omega_{\mathcal{PP}_4}|$.

#	$\mathcal{B} = \mathrm{Walsh}(\mathcal{U})$	type of $\mathcal{U}$	$\mathrm{Aut}(\mathcal{U})$	d_1	Υ	ι
9	$\mathrm{Walsh}(\mathcal{PP}_{2k})$	$(2,2,2k)$	D_{2k}	2	1	1
10	$\mathrm{Walsh}(\mathcal{PC})$	$(3,2,4)$	S_4	1	S_4	24
11	$\mathrm{Walsh}(\mathcal{PD})$	$(3,2,5)$	A_5	1	A_5	60
12	$\mathrm{Walsh}(\mathrm{D}_{(02)}(\mathcal{PC}))$	$(4,2,3)$	S_4	2	V_4	4
13	$\mathrm{Walsh}(\mathrm{D}_{(02)}(\mathcal{PD}))$	$(5,2,3)$	A_5	1	A_5	60
14	$\mathrm{Walsh}(\mathrm{D}_{(02)}(\mathcal{PP}_{4k}))$	$(4k,2,2)$	D_{4k}	2	D_{2k}	$4k$
	$\mathrm{Walsh}(\mathrm{D}_{(02)}(\mathcal{PP}_{4k-2}))$	$(4k-2,2,2)$	D_{4k-2}	2	C_{2k-1}	$2k-1$
15	$\mathrm{Walsh}(\mathrm{D}_{(12)}(\mathcal{PC}))$	$(3,4,2)$	S_4	1	S_4	24
16	$\mathrm{Walsh}(\mathrm{D}_{(12)}(\mathcal{PD}))$	$(3,5,2)$	A_5	1	A_5	60

Table 3.4: The bipartite-regular hypermaps obtained by the Walsh construction.

Chirality groups and chirality indices of $\mathcal{B} = \mathrm{Pin}(\mathcal{R})$

Now we assume that $\mathcal{R}$ is a regular hypermap on the projective plane of type (l, m, n) and $\mathcal{B} = \mathrm{Pin}(\mathcal{R})$. As before, $T = \{(R_1R_2)^l, (R_2R_0)^m, (R_0R_1)^n, A_{\mathcal{R}^+}\}$ and $S = \{A_{\mathcal{R}^+}\}$.

According to Table 3.3, up to duality, there are 8 types of bipartite-regular hypermaps on the projective plane obtained from regular hypermaps using the Pin construction.

If $d_2 = 1$, then, by Corollary 1.9.7, $\Upsilon(\mathcal{B}) = \Delta/R \cong \mathrm{Aut}(\mathcal{R})$ and $\mathcal{B}^\Delta$ is $\mathcal{T}_{\Delta\hat{0}}$.

Table 3.5 lists the 8 types of bipartite-regular hypermaps on the projective plane obtained from regular hypermaps using the Pin construction. Of those cases, only 3 are non-uniform with $d_2 \neq 1$: cases 1, 4 and 8. The chirality groups of these hypermaps are computed below. The last two columns of Table 3.5 display the chirality groups and chirality indices.

- Case **1**: $\mathcal{B} = \mathrm{Pin}(\mathcal{R})$, $\mathcal{R} = \mathcal{PP}_{2k}$ has type $(2, 2, 2k)$ and $d_2 = 2$. Then
  $A_{\mathcal{R}^+} = A_{\mathcal{P}_{2k}} = (R_0R_1)^kR_2$, $A_{\mathcal{R}^+}\alpha_P = (R_1R_0)^kR_0$,
  $\Upsilon(\mathcal{B}) \cong \langle RR_1R_2, R(R_0R_1)^2, R(R_1R_0)^kR_0 = RR_2R_0\rangle^{\Delta/R} = R\Delta^+/R = \Delta/R \cong D_{2k}$
  and $\iota = 4k$; $\mathcal{B}^\Delta$ is $\mathcal{T}_{\Delta\hat{0}}$.

- Case **4**: $\mathcal{B} = \mathrm{Pin}(\mathcal{R})$, $\mathcal{R} = \mathcal{PC}$ has type $(3, 2, 4)$ and $d_2 = 2$. Then
  $A_{\mathcal{R}^+} = A_{\mathcal{C}} = (R_0R_1R_2)^3$, $A_{\mathcal{R}^+}\alpha_P = (R_1R_0R_0)^3 = R_1$,
  $\Upsilon(\mathcal{B}) \cong \langle RR_1R_2, R(R_0R_1)^2, RR_1\rangle^{\Delta/R} = \langle RR_1, RR_2, RR_0\rangle^{\Delta/R} = \Delta/R \cong S_4$
  since $RR_0 = RR_1R_2{R_1}^{R_0}{R_2}^{R_0}R_1R_2$, and $\iota = 24$; $\mathcal{B}^\Delta$ is $\mathcal{T}_{\Delta\hat{0}}$.

- Case **8**: $\mathcal{B} = \mathrm{Pin}(\mathcal{R})$, $\mathcal{R} = \mathcal{PC}$ has type $(2k, 2, 2)$ and $d_2 = 2$. Then
  $A_{\mathcal{R}^+} = (R_2R_1)^kR_0$ (see Case 12), $A_{\mathcal{R}^+}\alpha_P = (R_0R_0)^kR_1 = R_1$,
  $\Upsilon(\mathcal{B}) \cong \langle RR_1R_2, R(R_0R_1)^2, RR_1\rangle^{\Delta/R} = \langle RR_1, RR_2, RR_0\rangle^{\Delta/R} = \Delta/R \cong D_{2k}$
  since $RR_0 = R(R_2R_1)^k$, and $\iota = 4k$; $\mathcal{B}^\Delta$ is $\mathcal{T}_{\Delta\hat{0}}$.

#	$\mathcal{B} = \mathrm{Pin}(\mathcal{U})$	type of $\mathcal{U}$	$\mathrm{Aut}(\mathcal{U})$	d_2	Υ	ι
1	$\mathrm{Pin}(\mathcal{PP}_{2k})$	$(2, 2, 2k)$	D_{2k}	2	D_{2k}	$4k$
2	$\mathrm{Pin}(\mathrm{D}_{(01)}(\mathcal{PC}))$	$(2, 3, 4)$	S_4	1	S_4	24
3	$\mathrm{Pin}(\mathrm{D}_{(01)}(\mathcal{PD}))$	$(2, 3, 5)$	A_5	1	A_5	60
4	$\mathrm{Pin}(\mathcal{PC})$	$(3, 2, 4)$	S_4	2	S_4	24
5	$\mathrm{Pin}(\mathcal{PD})$	$(3, 2, 5)$	A_5	1	A_5	60
6	$\mathrm{Pin}(\mathrm{D}_{(02)}(\mathcal{PC}))$	$(4, 2, 3)$	S_4	1	S_4	24
7	$\mathrm{Pin}(\mathrm{D}_{(02)}(\mathcal{PD}))$	$(5, 2, 3)$	A_5	1	A_5	60
8	$\mathrm{Pin}(\mathrm{D}_{(02)}(\mathcal{PP}_{2k}))$	$(2k, 2, 2)$	D_{2k}	2	D_{2k}	$4k$

Table 3.5: The bipartite-regular hypermaps obtained by the Pin construction.

The closure covers and the covering cores

Table 3.6 lists the chirality groups and chirality indices of all the bipartite-regular hypermaps on the on the sphere, as well as their closure covers. Table 3.7 displays the type, number of flags and genus of the covering cores.

In case 9 of Table 3.3, $\mathcal{B} = \mathrm{Walsh}(\mathcal{PP}_{2k})$ is uniform and hence regular, so $\mathcal{B} = \mathcal{B}^\Delta = \mathcal{B}_\Delta$.

According to Tables 3.4 and 3.5, in 13 out of the 16 cases, $\Upsilon(\mathcal{B}) = \Delta^{\hat{0}}/B \cong \Delta/R$ and $\mathcal{B}^{\Delta}$ is $\mathcal{T}_{\Delta\hat{0}}$. If $\mathcal{B}$ is Walsh($\mathrm{D}_{(02)}(\mathcal{PC})$), Walsh($\mathrm{D}_{(02)}(\mathcal{PP}_{4k-2})$) or Walsh($\mathrm{D}_{(02)}(\mathcal{PP}_{4k})$), then $\mathcal{B}^{\Delta}$ is $\mathcal{PP}_6$, $\mathcal{PP}_4$ or $\mathcal{T}_{\Delta\hat{0}\hat{1}\hat{2}}$, respectively.

According to Theorem 1.8.4, in cases 1, 4, 8 and 14 (with k even) of Table 3.7, $\mathcal{B}_{\Delta}$ is orientable because $|\Omega_{\mathcal{B}_{\Delta}}| = |\Omega_{(\mathcal{B}^{+})_{\Delta}}|$, and in the remaining cases $\mathcal{B}_{\Delta}$ is non-orientable since $|\Omega_{\mathcal{B}_{\Delta}}| < 2|\Omega_{\mathcal{B}_{\Delta}}| = |\Omega_{(\mathcal{B}^{+})_{\Delta}}|$.

The covering core of the map Walsh($\mathrm{D}_{(02)}(\mathcal{PC})$) is a non-orientable regular map of type $(4, 2, 6)$, with 192 flags, 48 edges and Petrie path of length 6. In [70, 69], Wilson denotes this map by $D(70)$. We remark that its orientable double cover is the hypermap denoted by $DP(190)$ in [70, 69], the closure cover of Walsh($\mathrm{D}_{(02)}(\mathcal{C})$) (case 17 of Table 2.7).

#	$\mathcal{B}$	$\|\Omega_{\mathcal{B}}\|$	$\mathcal{B}^{\Delta}$	type of $\mathcal{B}^{\Delta}$	$\|\Omega_{\mathcal{B}^{\Delta}}\|$	Υ	ι
1	Pin($\mathcal{PP}_{2k}$)	$8k$	$\mathcal{T}_{\Delta\hat{0}}$	–	2	D_{2k}	$4k$
2	Pin($\mathrm{D}_{(01)}(\mathcal{PC})$)	48	$\mathcal{T}_{\Delta\hat{0}}$	–	2	S_4	24
3	Pin($\mathrm{D}_{(01)}(\mathcal{PD})$)	120	$\mathcal{T}_{\Delta\hat{0}}$	–	2	A_5	60
4	Pin($\mathcal{PC}$)	48	$\mathcal{T}_{\Delta\hat{0}}$	–	2	S_4	24
5	Pin($\mathcal{PD}$)	120	$\mathcal{T}_{\Delta\hat{0}}$	–	2	A_5	60
6	Pin($\mathrm{D}_{(02)}(\mathcal{PC})$)	48	$\mathcal{T}_{\Delta\hat{0}}$	–	2	S_4	24
7	Pin($\mathrm{D}_{(02)}(\mathcal{PD})$)	120	$\mathcal{T}_{\Delta\hat{0}}$	–	2	A_5	60
8	Pin($\mathrm{D}_{(02)}(\mathcal{PP}_{2k})$)	$8k$	$\mathcal{T}_{\Delta\hat{0}}$	–	2	D_{2k}	$4k$
9	Walsh($\mathcal{PP}_{2k}$)	$8k$	$\mathcal{PP}_{4k}$	$(2, 2, 4k)$	$8k$	1	1
10	Walsh($\mathcal{PC}$)	48	$\mathcal{T}_{\Delta\hat{0}}$	–	2	S_4	24
11	Walsh($\mathcal{PD}$)	120	$\mathcal{T}_{\Delta\hat{0}}$	–	2	A_5	60
12	Walsh($\mathrm{D}_{(02)}(\mathcal{PC})$)	48	$\mathcal{PP}_6$	$(2, 2, 6)$	12	V_4	4
13	Walsh($\mathrm{D}_{(02)}(\mathcal{PD})$)	120	$\mathcal{T}_{\Delta\hat{0}}$	–	2	A_5	60
14	Walsh($\mathrm{D}_{(02)}(\mathcal{PP}_{4k})$)	$16k$	$\mathcal{T}_{\Delta\hat{0}\hat{1}\hat{2}}$	–	4	D_{2k}	$4k$
	Walsh($\mathrm{D}_{(02)}(\mathcal{PP}_{4k-2})$)	$16k-8$	$\mathcal{PP}_4$	$(2, 2, 4)$	8	C_{2k-1}	$2k-1$
15	Walsh($\mathrm{D}_{(12)}(\mathcal{PC})$)	48	$\mathcal{T}_{\Delta\hat{0}}$	–	2	S_4	24
16	Walsh($\mathrm{D}_{(12)}(\mathcal{PD})$)	120	$\mathcal{T}_{\Delta\hat{0}}$	–	2	A_5	60

Table 3.6: $\mathcal{B}$ and $\mathcal{B}^{\Delta}$

#	$\mathcal{B}$	type of $\mathcal{B}_\Delta$	$\lvert\Omega_{\mathcal{B}_\Delta}\rvert$	or.?	genus	Υ	ι
1	Pin$(\mathcal{PP}_{2k})$	$(2,4,4k)$	$32k^2$	yes	$2k^2-2k+1$	D_{2k}	$4k$
2	Pin$(\mathrm{D}_{(01)}(\mathcal{PC}))$	$(2,6,8)$	1152	no	122	S_4	24
3	Pin$(\mathrm{D}_{(01)}(\mathcal{PD}))$	$(2,6,10)$	7200	no	842	A_5	60
4	Pin$(\mathcal{PC})$	$(3,4,8)$	1152	yes	85	S_4	24
5	Pin$(\mathcal{PD})$	$(3,4,10)$	7200	no	1142	A_5	60
6	Pin$(\mathrm{D}_{(02)}(\mathcal{PC}))$	$(4,4,6)$	1152	no	194	S_4	24
7	Pin$(\mathrm{D}_{(02)}(\mathcal{PD}))$	$(5,4,6)$	7200	no	1382	A_5	60
8	Pin$(\mathrm{D}_{(02)}(\mathcal{PP}_{2k}))$	$(2k,4,4)$	$32k^2$	yes	$(2k-1)^2$	D_{2k}	$4k$
9	Walsh$(\mathcal{PP}_{2k})$	$(2,2,4k)$	$8k$	no	1	1	1
10	Walsh$(\mathcal{PC})$	$(6,2,8)$	1152	no	122	S_4	24
11	Walsh$(\mathcal{PD})$	$(6,2,10)$	7200	no	842	A_5	60
12	Walsh$(\mathrm{D}_{(02)}(\mathcal{PC}))$	$(4,2,6)$	192	no	10	V_4	4
13	Walsh$(\mathrm{D}_{(02)}(\mathcal{PD}))$	$(10,2,6)$	7200	no	842	A_5	60
14	Walsh$(\mathrm{D}_{(02)}(\mathcal{PP}_{4k}))$	$(4k,2,4)$	$64k^2$	yes	$(2k-1)^2$	D_{2k}	$4k$
	Walsh$(\mathrm{D}_{(02)}(\mathcal{PP}_{4k-2}))$	$(4k-2,2,4)$	$8(2k-1)^2$	no	$2(k-1)^2+1$	C_{2k-1}	$2k-1$
15	Walsh$(\mathrm{D}_{(12)}(\mathcal{PC}))$	$(12,2,4)$	1152	no	98	S_4	24
16	Walsh$(\mathrm{D}_{(12)}(\mathcal{PD}))$	$(15,2,4)$	7200	no	662	A_5	60

Table 3.7: $\mathcal{B}$ and $\mathcal{B}_\Delta$.

Chapter 4

Hypermaps on the torus

Up to duality, there are 3 possibilities for the type of a uniform hypermap on the torus: $(4,2,4)$, $(6,2,3)$ and $(3,3,3)$. In [33], Coxeter and Moser classify the orientably-regular maps on the torus: orientably-regular hypermaps of type $(4,2,4)$ can be represented by identifying opposite edges of a square with vertices in the lattice $\mathbb{Z}[i]$ and orientably-regular hypermaps of type $(6,2,3)$ can be represented by identifying opposite edges of a lozenge whose angles are $\pi/3$ and $2\pi/3$ (that is, a lozenge that can be divided in 2 equilateral triangles) with vertices in the lattice $\mathbb{Z}[\rho]$, where $\rho = (1+\sqrt{3}i)/2$. They also gave conditions for an orientably-regular map to be regular. Corn and Singerman [28] proved that a uniform hypermap $\mathcal{U}$ of type $(3,3,3)$ is orientably-regular if and only if Walsh($\mathcal{U}$) is orientably-regular. More recently, Breda and Nedela [10] have shown that $\mathcal{U}$ is orientably-chiral if and only if Walsh($\mathcal{U}$) is orientably-chiral. Consequently, $\mathcal{U}$ is regular if and only if Walsh($\mathcal{U}$) is regular. In [57] and [58], Singerman and Syddall classify the uniform maps on the torus.

Širáň, Tucker and Watkins [66] studied the edge-transitive maps on the torus, which include, up to duality, all 2-restrictedly-regular hypermaps on the torus except the $\Delta^{\hat{1}}$-chiral. The correspondence between the types of edge-transitive maps of Graver and Watkins [36] used by Širáň, Tucker and Watkins [66] and restrictedly-regular maps is given in Table 1.1.

As mentioned in Chapter 1, $\Delta^{\hat{0}1\hat{2}}$-chiral maps were called just-edge-transitive maps by Jones [47] and edge-transitive maps of type 3 by Graver and Watkins [36]. Their automorphism group acts transitively on edges but neither on vertices nor faces.

In this Chapter we introduce a notation for uniform hypermaps on the torus which we use in the classification of the regular and the 2-restrictedly-regular hypermaps on the torus. This notation is based in the work of Singerman and Syddall (see [57] and [58]) on uniform maps on the torus and extends the notation of Coxeter and Moser [33] for orientably-regular hypermaps.

The results in this Chapter were obtained before knowing the work of Širáň, Tucker and Watkins on edge-transitive maps on the torus [66].

4.1 Uniform hypermaps on the torus

Let $\mathcal{U}$ be a uniform hypermap on the torus of type (l,m,n). Using the Euler formula for uniform hypermaps (Corollary 1.4.2) together with Lemma 1.4.5, one can see that, up to duality, (l,m,n) is $(2,4,4)$, $(2,3,6)$ or $(3,3,3)$.

The uniform maps on the torus of types $(4,2,4)$ and $(6,2,3)$ were classified by Singerman and Syddall in [57, 58]. These maps are obtained by identifying the opposite edges of a Euclidean parallelogram in the complex plane with vertices in the lattices $\mathbb{Z}[i]$ or $\mathbb{Z}[\rho]$, where $\rho = (1+\sqrt{3}i)/2 = e^{i\pi/3}$.

Let $a,b,c,d \in \mathbb{Z}$, $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and $\alpha \in \{i,\rho\}$. The complexes 0, $a+b\alpha$ and $c+d\alpha$ are in the same straight line if and only if $\det(M) = ad-bc = 0$. Thus, 0, $a+b\alpha$, $c+d\alpha$ and $(a+b\alpha)+(c+d\alpha)$ are the vertices of a Euclidean parallelogram in the complex plane if and only if $\det(M) = ad-bc \neq 0$.

Instead of the notation used by Singerman and Syddall in [58], we adopt a notation which is a natural extension of the notation of Coxeter and Moser [33] for orientably-regular maps on the torus. Given $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ such that $\det(M) = ad-bc \neq 0$, we denote by $(4,2,4)_M$ and $(6,2,3)_M$ the uniform maps on the torus of types $(4,2,4)$ and $(6,2,3)$ represented by the Euclidean parallelograms with opposite edges identified and vertices 0, $a+b\alpha$, $c+d\alpha$, $(a+c)+(b+d)\alpha \in \mathbb{Z}[\alpha]$, where $\alpha = i$ in the first case, and $\alpha = \rho$ in the second case. In particular, the maps denoted by $\{4,4\}_{p,q}$ and $\{3,6\}_{p,q}$ in the notation of Coxeter and Moser are denoted by $(4,2,4)_{\left(\begin{smallmatrix} p & -q \\ q & p \end{smallmatrix}\right)}$ and $(6,2,3)_{\left(\begin{smallmatrix} p & -p-q \\ q & p \end{smallmatrix}\right)}$, respectively. Figure 4.1 displays $(4,2,4)_M$ and $(6,2,3)_M$, for $M = \begin{pmatrix} 1 & -3 \\ 2 & 2 \end{pmatrix}$.

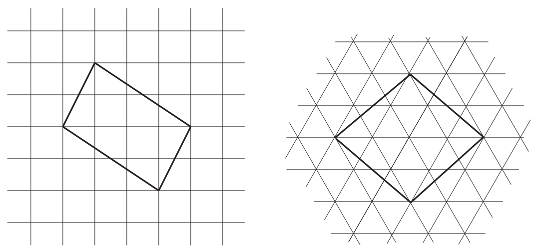

Figure 4.1: The hypermaps $(4,2,4)_M$ and $(6,2,3)_M$ for $M = \left(\begin{smallmatrix} 1 & -3 \\ 2 & 2 \end{smallmatrix}\right)$.

Let $\mathcal{U} = (4,2,4)_{\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)}$, $a+bi = (p+qi)(r+si)$ and $c+di = (p+qi)(t+ui)$, where $r+si$ and $t+ui$ are coprime Gaussian integers. In the notation of Singerman and Syddall $\mathcal{U}$ is denoted by $\left\{\frac{t+ui}{r+si}\right\}_{p+qi}$ or $\left\{\frac{r+si}{t+ui}\right\}_{p+qi}$, depending on whether $\frac{t+ui}{r+si}$ has positive or negative imaginary part, or equivalently, depending on whether $ad-bc$ is positive or negative. Conversely, if $\mathcal{U}$ is denoted by $\left\{\frac{t+ui}{r+si}\right\}_{p+qi}$ in the notation of Singerman and Syddall, then $\mathcal{U}$ is $(4,2,4)_{\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)}$, where $a+bi = (p+qi)(r+si)$ and $c+di = (p+qi)(t+ui)$. Similarly, there is a correspondence between our notation and the notation of Singerman and Syddall for uniform maps on the torus of type $(6,2,3)$.

The hypermaps $(4,2,4)_{\left(\begin{smallmatrix} 2 & -1 \\ 1 & 2 \end{smallmatrix}\right)}$, $(6,2,3)_{\left(\begin{smallmatrix} 2 & -1 \\ 1 & 3 \end{smallmatrix}\right)}$ and $\mathrm{D}_{(02)}\left((6,2,3)_{\left(\begin{smallmatrix} 1 & -1 \\ 1 & 2 \end{smallmatrix}\right)}\right)$ (denoted by $\{i\}_{2+i}$, $\{\rho\}_{2+\rho}$ and $\mathrm{D}_{(02)}(\{\rho\}_{1+\rho})$ in the notation Singerman and Syddall) are uniform imbeddings of the non-planar graphs K_5, K_7 and $K_{3,3}$ (see §8.3 and §8.4 of [33] and §5 of [58]). Using the Euler formula, one can see that there is no uniform imbedding of K_6 on the torus, that is, there is no uniform map on the torus whose underlying graph is K_6. For otherwise, such imbedding would have 6 vertices, 15 edges and $f = e - v = 9$ faces and 60 flags but $18 = 2f \nmid 60$.

Throughout this chapter we assume that $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, M' = \begin{pmatrix} a' & c' \\ b' & d' \end{pmatrix} \in M(2, \mathbb{Z})$, with $\det(M) \neq 0 \neq \det(M')$.

4.1.1 Uniform maps on the torus of types $(4, 2, 4)$ and $(6, 2, 3)$.

Let $N_4 := N(4, 2, 4)$ and $N_6 := N(6, 2, 3)$, where, as before, $N(l, m, n)$ is the normal closure of the subgroup generated by $(R_1R_2)^l$, $(R_2R_0)^m$ and $(R_0R_1)^n$ in Δ.

Lemma 4.1.1. *1. The hypermap $\mathcal{U} = (4, 2, 4)_M$ has $F = |\det(M)|$ faces, $E = 2|\det(M)|$ edges, $V = |\det(M)|$ vertices and $|\Omega_{\mathcal{U}}| = 8|\det(M)|$ flags;*

2. The hypermap $\mathcal{U} = (6, 2, 3)_M$ has $F = 2|\det(M)|$ faces, $E = 3|\det(M)|$ edges, $V = |\det(M)|$ vertices and $|\Omega_{\mathcal{U}}| = 12|\det(M)|$ flags.

Proof. Let $\alpha \in \{i, \rho\}$. In both cases all faces of $\mathcal{U}$ are represented by congruent polygons. Thus, the area of the Euclidean parallelogram with vertices 0, $a + b\alpha$, $c + d\alpha$ and $(a + c) + (b + d)\alpha$, is equal to the number of faces F times the area of one face. On the other hand, the area of the Euclidean parallelogram is $|\det(C \cdot M)| = |\det(C)| \cdot |\det(M)|$, where $C = \begin{pmatrix} 1 & \mathrm{Re}(\alpha) \\ 0 & \mathrm{Im}(\alpha) \end{pmatrix}$ is the matrix of change of basis from $(1, \alpha)$ to $(1, i)$. When $\alpha = i$, the faces are represented by squares with area 1 and $\det(C) = 1$, so $|\det(M)| = F \times 1 = F$. When $\alpha = \rho$, the faces are represented by equilateral triangles with area $\sqrt{3}/4$ and $\det(C) = \sqrt{3}/2$, so $F = 2|\det(M)|$. The other values are given by the formula $|\Omega_{\mathcal{U}}| = 2lV = 2mE = 2nF$. □

Let $X_4 = R_0R_1R_2R_1$, $Y_4 = {X_4}^{R_1} = R_1R_0R_1R_2$, $X_6 = R_0R_1R_2R_1R_2R_1$ and $Y_6 = {X_6}^{R_1} = R_1R_0R_1R_2R_1R_2$. We shall omit the index l in N_l, X_l and Y_l if it is clear from the context.

Lemma 4.1.2 (Properties of N, X and Y)**.**

1. *If $N = N_4$, $X = X_4$ and $Y = Y_4$, then:*
   (a) $NX \leftrightharpoons NY$;
   (b) $NX^{R_0} = NX^{-1}$, $NX^{R_1} = NY$, $NX^{R_2} = NX$,
   $NY^{R_0} = NY$, $NY^{R_1} = NX$, $NY^{R_2} = NY^{-1}$.

2. *If $N = N_6$, $X = X_6$ and $Y = Y_6$, then:*
   (a) $NX \leftrightharpoons NY$;
   (b) $NX^{R_0} = NX^{-1}$, $NX^{R_1} = NY$, $NX^{R_2} = NX$,
   $NY^{R_0} = NX^{-1}Y$, $NY^{R_1} = NX$, $NY^{R_2} = NXY^{-1}$.

Proof. 1.(a) $NXNYNX^{-1}NY^{-1} = NXYX^{-1}Y^{-1}$

$$= N\left[(R_0R_1)^4\left((R_2R_0)^{-2}\right)^{R_1}(R_1R_2)^4(R_2R_0)^2\right]^{(R_0R_1)^2} = N.$$

(b)
$$\begin{aligned}
NX^{R_0} &= NR_1R_2R_1R_0 = NX^{-1},\\
NX^{R_1} &= NY,\\
NX^{R_2} &= N(R_0R_1R_2R_1)^{R_2} = N\left[(R_2R_0)^2(R_1R_2)^4\right]^{R_2R_0} X = NX\\
NY^{R_0} &= N(R_1R_0R_1R_2)^{R_0} = N\left[(R_0R_1)^4(R_2R_0)^2\right]^{(R_0R_1)^2} Y = NY\\
NY^{R_1} &= N(X^{R_1})^{R_1} = NX,\\
NY^{R_2} &= NR_2R_1R_0R_1 = NY^{-1}.
\end{aligned}$$

2.(a) $NXNYNX^{-1}NY^{-1} = NXYX^{-1}Y^{-1}$
$$= N\left[(R_1R_0)^3(R_0R_2)^2\left[(R_0R_1)^3\left((R_0R_2)^2\right)^{R_1}(R_1R_2)^6(R_2R_0)^2\right]^{R_0R_1R_2}\right]^{R_1R_0R_1} = N.$$

(b) $NX^{R_0} = NR_1R_2R_1R_2R_1R_0 = NX^{-1}$,
$NX^{R_1} = NY$,
$NX^{R_2} = N(R_0R_1R_2R_1R_2R_1)^{R_2} = N(R_2R_0)^2\left[(R_1R_2)^6\right]^{R_1R_0}X = NX$,
$NY^{R_0} = N(R_1R_0R_1R_2R_1R_2)^{R_0} = N(R_0R_1)^3\left[(R_1R_2)^{-6}\right]^{R_0R_1}YX^{-1} = NX^{-1}Y$,
$NY^{R_1} = N(X^{R_1})^{R_1} = NX$,
$NY^{R_2} = NR_2R_1R_0R_1R_2R_1 = N\left[(R_2R_0)^2(R_0R_1)^3\right]^{R_2R_0R_2}Y^{-1}X = NXY^{-1}$.

□

Remark 4.1.3. $NX^{R_2R_0} = NX^{-1}$ and $NY^{R_2R_0} = NY^{-1}$.

Because N is a normal subgroup of Δ contained in U, N is contained in U_Δ, and hence:

Corollary 4.1.4. *Let $\mathcal{U}$ be a uniform map on the torus and U a hypermap subgroup of $\mathcal{U}$.*

1. *If $\mathcal{U}$ is of type $(4,2,4)$, $X = X_4$ and $Y = Y_4$, then:*
   (a) $U_\Delta X \leftrightharpoons U_\Delta Y$;
   (b) $U_\Delta X^{R_0} = U_\Delta X^{-1}$, $U_\Delta X^{R_1} = U_\Delta Y$, $U_\Delta X^{R_2} = U_\Delta X$,
   $U_\Delta Y^{R_0} = U_\Delta Y$, $U_\Delta Y^{R_1} = U_\Delta X$, $U_\Delta Y^{R_2} = U_\Delta Y^{-1}$.
2. *If $\mathcal{U}$ is of type $(6,2,3)$, $X = X_6$ and $Y = Y_6$, then:*
   (a) $U_\Delta X \leftrightharpoons U_\Delta Y$;
   (b) $U_\Delta X^{R_0} = U_\Delta X^{-1}$, $U_\Delta X^{R_1} = U_\Delta Y$, $U_\Delta X^{R_2} = U_\Delta X$,
   $U_\Delta Y^{R_0} = U_\Delta X^{-1}Y$, $U_\Delta Y^{R_1} = U_\Delta X$, $U_\Delta Y^{R_2} = U_\Delta XY^{-1}$.

Remark 4.1.5. $U_\Delta X^{R_2R_0} = U_\Delta X^{-1}$ and $U_\Delta Y^{R_2R_0} = U_\Delta Y^{-1}$.

Now we use the previous results to obtain hypermaps subgroups for the uniform maps on the torus.

Theorem 4.1.6 (Hypermap subgroups of $(4,2,4)_M$ and $(6,2,3)_M$)**.**

1. $U = N\langle X^aY^b, X^cY^d\rangle = \langle (R_1R_2)^4, (R_2R_0)^2, (R_0,R_1)^4\rangle^\Delta\langle X^aY^b, X^cY^d\rangle$ *is a hypermap subgroup of* $\mathcal{U} = (4,2,4)_M$;
2. $U = N\langle X^aY^b, X^cY^d\rangle = \langle (R_1R_2)^6, (R_2R_0)^2, (R_0,R_1)^3\rangle^\Delta\langle X^aY^b, X^cY^d\rangle$ *is a hypermap subgroup of* $\mathcal{U} = (6,2,3)_M$.

Proof. By the definition of $\mathcal{U}$, $N \subseteq U$ and $X^aY^b, X^cY^d \in U$. Let $V := N\langle X^aY^b, X^cY^d\rangle$. Since $N \lhd \Delta$, V is a subgroup of Δ such that $N \subseteq V \subseteq U$. Furthermore V/N is a subgroup of Δ/N. By Lemma 4.1.2, for all $p,q \in \mathbb{Z}$, $NX^pY^q = (NX)^p(NX)^q$, so $V/N \cong \langle (a,b),(c,d)\rangle$. Therefore V/N has index $2l|ad-bc| = 2l|\det(M)| = |\Omega_\mathcal{U}| = [\Delta : U]$ in Δ/N. It follows that $[\Delta : U] = [\Delta/N : V/N] = [\Delta : V] = [\Delta : U][U : V]$, so $[U : V] = 1$, that is, $U = V$. □

Using Theorem 4.1.6 together with Remark 4.1.5, we get:

Corollary 4.1.7.

1. $U/N = \langle NX^aY^b, NX^cY^d\rangle \cong \langle NX^aY^b\rangle \times \langle NX^cY^d\rangle \cong \mathbb{Z}\times\mathbb{Z}$;
2. $U^{R_2R_0} = U$, *or equivalently,* $U^{R_0} = U^{R_2}$.

Proposition 4.1.8 (Conservativeness of $(4,2,4)_M$ and $(6,2,3)_M$). *Let* $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ *with* $\det(M) \neq 0$. *Then:*

1. (a) $(4,2,4)_M$ *is* Δ^+-, $\Delta^{\hat{1}}$-, Δ^1-*conservative;*
   (b) $(4,2,4)_M$ *is* $\Delta^{\hat{0}}$-, $\Delta^{\hat{2}}$-, Δ^0-, Δ^2-*conservative if and only if* $a-b$ *and* $c-d$ *are even.*
2. (a) $(6,2,3)_M$ *is* Δ^+-, $\Delta^{\hat{2}}$-, Δ^2-*conservative;*
   (b) $(6,2,3)_M$ *is not* $\Delta^{\hat{0}}$-, $\Delta^{\hat{1}}$-, Δ^0-, Δ^1-*conservative.*

Proof. 1. Clearly, $(R_1R_2)^4$, $(R_2R_0)^2$ and $(R_0R_1)^4$ are in every normal subgroup of index 2 of Δ. Thus $N \subseteq \Theta$, for all $\Theta \lhd_2 \Delta$. Owing to this, $U \subseteq \Theta$ if and only if $X^aY^b, X^cY^d \in \Theta$. Because $X, Y \in \Delta^+, \Delta^{\hat{1}}, \Delta^1$ but $X, Y \notin \Delta^{\hat{0}}, \Delta^{\hat{2}}, \Delta^0, \Delta^2$, it follows that $X^pY^q \in \Delta^+, \Delta^{\hat{1}}, \Delta^1$ for every $p, q \in \mathbb{Z}$, and $X^pY^q \in \Delta^{\hat{0}}, \Delta^{\hat{2}}, \Delta^0, \Delta^2$ if and only if p and q are both even or both odd, or equivalently, if $p-q$ is even.
2. $(R_1R_2)^6$ and $(R_2R_0)^2$ are in every subgroup of index 2 of Δ, but $(R_0R_1)^3$ is only in Δ^+, $\Delta^{\hat{2}}$ and Δ^2. Because of this, $\mathcal{U}$ can never be $\Delta^{\hat{0}}$-, $\Delta^{\hat{1}}$-, Δ^0-, Δ^1-conservative. Since X and Y are in Δ^+, $\Delta^{\hat{2}}$ and Δ^2, $\mathcal{U}$ is always Δ^+-, $\Delta^{\hat{2}}$-, Δ^2-conservative. □

Lemma 4.1.9. $\Delta/U_\Delta = \langle U_\Delta R_1, U_\Delta R_2\rangle\langle U_\Delta X, U_\Delta Y\rangle$. *In addition,* $\langle U_\Delta R_1, U_\Delta R_2\rangle \cong D_l$ *and* $\langle U_\Delta X, U_\Delta Y\rangle$ *is abelian.*

Proof. Let $S = \langle U_\Delta R_1, U_\Delta R_2\rangle = U_\Delta\langle R_1, R_2\rangle/U_\Delta$ and $T = \langle U_\Delta X, U_\Delta Y\rangle = U_\Delta\langle X, Y\rangle/U_\Delta$. Using Corollary 4.1.4, we have that $(U_\Delta R_i)T = T(U_\Delta R_i)$, for $i \in \{1,2\}$, so $ST = TS$. It follows that ST is a subgroup of Δ/U_Δ containing $U_\Delta R_0 = U_\Delta R_1R_2R_1X^{-1}$, $U_\Delta R_1$ and $U_\Delta R_2$, so $\Delta/U_\Delta = \langle U_\Delta R_0, U_\Delta R_1, U_\Delta R_2\rangle \subseteq ST \subseteq \Delta/U_\Delta$, that is, $\Delta/U_\Delta = ST$. □

In [57, 58] Singerman and Syddall determined conditions for seeing if two given uniform maps on the torus of the same type are isomorphic or not. However they did not see when one covers the other. Our next result fills this gap.

Theorem 4.1.10.

1. (a) $(4,2,4)_M \to (4,2,4)_{M'}$ *if and only if there are* $P \in \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle \cong D_4$ *and* $Q \in M(2,\mathbb{Z})$ *such that* $\det(Q) \neq 0$ *and* $M = PM'Q$*;*
   (b) $(4,2,4)_M \cong (4,2,4)_{M'}$ *if and only if there are* $P \in \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle \cong D_4$ *and* $Q \in GL(2,\mathbb{Z})$ *such that* $M = PM'Q$.
2. (a) $(6,2,3)_M \to (6,2,3)_{M'}$ *if and only if there are* $P \in \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \right\rangle \cong D_6$ *and* $Q \in M(2,\mathbb{Z})$ *such that* $\det(Q) \neq 0$ *and* $M = PM'Q$.
   (b) $(6,2,3)_M \cong (6,2,3)_{M'}$ *if and only if there are* $P \in \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \right\rangle \cong D_6$ *and* $Q \in GL(2,\mathbb{Z})$ *such that* $M = PM'Q$.

Proof. Let $\mathcal{U} = (l,m,n)_M, \mathcal{U}' = (l,m,n)_{M'}, U = N\langle X^aY^b, X^cY^d\rangle$ and $U' = N\langle X^{a'}Y^{b'}, X^{c'}Y^{d'}\rangle$. Furthermore, let $A_4 = A_6 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $B_4 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $B_6 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$. Equivalently, $A_l = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $B_l = \left(\begin{smallmatrix} 1 & (l-4)/2 \\ 0 & -1 \end{smallmatrix}\right)$.

Proof of 1(a) and 2(a):

($\Rightarrow$) Suppose that $\mathcal{U}$ covers $\mathcal{U}'$. Then, according to Lemma 1.3.1, there is $g \in \Delta$ such that $U \subseteq (U')^g$. By Lemma 4.1.9, $U'_\Delta g = (U'_\Delta s)(U'_\Delta t)$, for some $U'_\Delta s \in S = \langle U'_\Delta R_1, U'_\Delta R_2\rangle$ and $U'_\Delta t \in T = \langle U'_\Delta X, U'_\Delta Y\rangle$. Since $U'_\Delta t \in T = \langle U'_\Delta X, U'_\Delta Y\rangle$ and T is abelian (see Lemma 4.1.9), $U'_\Delta t \leftrightharpoons U'_\Delta X, U'_\Delta Y$. If $X^uY^v \in U \subseteq (U')^g$, then:

$$\begin{aligned} U' &= U'gX^uY^vg^{-1} = U'U'_\Delta g U'_\Delta (X^uY^v) U'_\Delta g^{-1} \\ &= U'U'_\Delta s U'_\Delta t U'_\Delta (X^uY^v) U'_\Delta t^{-1} U'_\Delta s^{-1} = U'U'_\Delta s U'_\Delta (X^uY^v) U'_\Delta s^{-1} \\ &= U'(U'_\Delta X^uY^v)^{U'_\Delta s^{-1}} \end{aligned}$$

Because $X^aY^b, X^cY^d \in U \subseteq (U')^g$, $U'(U'_\Delta X^aY^b)^{U'_\Delta s^{-1}} = U' = U'(U'_\Delta X^cY^d)^{U'_\Delta s^{-1}}$.

Clearly, $(U'_\Delta R_1)^2 = (U'_\Delta R_2)^2 = (U'_\Delta (R_1R_2))^l = U'_\Delta$. Let $\lambda_l : S = \langle U'_\Delta R_1, U'_\Delta R_2\rangle \to \langle A_l, B_l\rangle \cong D_l$ be the group isomorphism defined by $U'_\Delta R_1\lambda_l = A_l{}^{-1}$ and $U'_\Delta R_2\lambda_l = B_l{}^{-1}$. Then $(U'_\Delta X^uY^v)^{U'_\Delta R_1} = U'_\Delta X^{u_1}Y^{v_1}$ and $(U'_\Delta X^uY^v)^{U'_\Delta R_2} = U'_\Delta X^{u_2}Y^{v_2}$, where u_1, v_1, u_2, v_2 are given by

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = A_l \begin{pmatrix} u \\ v \end{pmatrix} = (U'_\Delta R_1\lambda_l)^{-1}\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = B_l \begin{pmatrix} u \\ v \end{pmatrix} = (U'_\Delta R_2\lambda_l)^{-1}\begin{pmatrix} u \\ v \end{pmatrix}.$$

We remark that D_l acts on $\mathbb{Z} \times \mathbb{Z}$ by right multiplication.

Thus, for all $u, v \in \mathbb{Z}$, and for all $U'_\Delta r \in \langle U'_\Delta X, U'_\Delta Y\rangle$, $(U'_\Delta X^uY^v)^{U'_\Delta r} = U'_\Delta X^{u^*}Y^{v^*}$, where u^* and v^* are given by $\begin{pmatrix} u^* \\ v^* \end{pmatrix} = (U'_\Delta r\lambda_l)^{-1}\begin{pmatrix} u \\ v \end{pmatrix}$.

Let $P = U'_\Delta s^{-1}\lambda_l$, and let $a^*, b^*, c^*, d^* \in \mathbb{Z}$ such that $U_\Delta X^{a^*}Y^{b^*} = (U_\Delta X^aY^b)^{U_\Delta s^{-1}}$ and $U_\Delta X^{c^*}Y^{d^*} = (U_\Delta X^cY^d)^{U_\Delta s^{-1}}$. Then:

$$\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = P^{-1}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = P^{-1}M.$$

Since $X^{a^*}Y^{b^*}, X^{c^*}Y^{d^*} \in U'$ and $U'/N \cong \langle NX^{a'}Y^{b'}\rangle \times \langle NX^{c'}Y^{d'}\rangle \cong \mathbb{Z}\times\mathbb{Z}$ (Corollary 4.1.7), there are $r, s, t, u \in \mathbb{Z}$ such that $NX^{a^*}Y^{b^*} = N(X^{a'}Y^{b'})^r N(X^{c'}Y^{d'})^s = NX^{a'r+c's}Y^{b'r+d's}$ and $NX^{c^*}Y^{d^*} = N(X^{a'}Y^{b'})^t N(X^{c'}Y^{d'})^u = NX^{a't+c'u}Y^{b't+d'u}$. Making $Q = \left(\begin{smallmatrix} r & t \\ s & u \end{smallmatrix}\right)$, it follows that

$$\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} a' & c' \\ b' & d' \end{pmatrix}\begin{pmatrix} r & t \\ s & u \end{pmatrix} = M'Q.$$

Hence $P^{-1}M = M'Q$ and $M = PP^{-1}M = PM'Q$.

($\Leftarrow$) Reciprocally, by choosing $r \in \langle R_1, R_2\rangle$ such that $U'_\Delta r\lambda_l = P^{-1}$, $U \subseteq (U')^r$ and $\psi : \Delta/_r U \to \Delta/_r U'$, $Ug \mapsto U'rg$ is a covering $\mathcal{U} \to \mathcal{U}'$.

Proof of 1(b) and 2(b):

($\Rightarrow$) If $\mathcal{U} \cong \mathcal{U}'$, then $\mathcal{U} \to \mathcal{U}'$ and $\mathcal{U}' \to \mathcal{U}$. By (a), there are $P_1, P_2 \in \langle A_l, B_l\rangle \cong D_l$ and $Q_1, Q_2 \in M(2,\mathbb{Z})$ such that $\det(Q_1), \det(Q_2) \neq 0$, $M = P_1M'Q_1$ and $M' = P_2MQ_2$. Owing to this $M = P_1P_2MQ_2Q_1$ and $\det(M) = \det(P_1)\cdot\det(P_2)\cdot\det(M)\cdot\det(Q_2)\cdot\det(Q_1)$. Since $\det(M) \neq 0$, $\det(P_1), \det(P_2), \det(Q_1), \det(Q_2) \in \{\pm 1\}$, so Q_1 and Q_2 are invertible.

($\Leftarrow$) Conversely, if there are $P \in \langle A_l, B_l \rangle \cong D_l$ and $Q \in M(2,\mathbb{Z})$ such that $\det(Q) \neq 0$ and $M = PM'Q$, then, by (a), $\mathcal{U} \to \mathcal{U}'$. Since P and Q are invertible, $M' = P^{-1}MQ^{-1}$ and, by (a), $\mathcal{U}' \to \mathcal{U}$. $\square$

Given $l \in \{4,6\}$, let $\sim_l$ be the binary relation defined on the set $\{M \in M(2,\mathbb{Z}) \mid \det(M) \neq 0\}$ by $M \sim_l M'$ if and only if there are $P \in \langle A_l, B_l \rangle \cong D_l$ and $Q \in GL(2,\mathbb{Z})$ such that $M = PM'Q$. Then $\sim_4$ and $\sim_6$ are equivalence relations such that

$$M \sim_4 M' \text{ if and only if } (4,2,4)_M \cong (4,2,4)_{M'}$$

and

$$M \sim_6 M' \text{ if and only if } (6,2,3)_M \cong (6,2,3)_{M'}.$$

In other words, Theorem 4.1.10 establishes a bijective correspondence between the equivalence classes of $\sim_4$ and $\sim_6$, and the isomorphism classes of uniform maps on the torus of types $(4,2,4)$ and $(6,2,3)$, respectively.

Corollary 4.1.11. $\mathrm{D}_{(02)}((4,2,4)_M) \cong (4,2,4)_M$.

Proof. Let $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. Then $(4,2,4)_M$ has hypermap subgroup $N\langle X^aY^b, X^cY^d \rangle$. Because $X\overline{(02)} = Y^{-1}$ and $Y\overline{(02)} = X^{-1}$, $(N\langle X^aY^b, X^cY^d \rangle)\overline{(02)} = N\langle X^{-b}Y^{-a}, X^{-d}Y^{-c} \rangle$ is a hypermap subgroup of $\mathrm{D}_{(02)}((4,2,4)_M)$. Since $\begin{pmatrix} -b & -d \\ -a & -c \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, the hypermaps are isomorphic by Theorem 4.1.10. $\square$

Now we give some examples of restrictedly-regular uniform maps on the torus:

Proposition 4.1.12. *Let $k, l, m \in \mathbb{Z}$.*

1. *(a)* $(4,2,4)_{\begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix}}$ *and* $(4,2,4)_{\begin{pmatrix} l & -l \\ m & m \end{pmatrix}}$ *are $\Delta^{\hat{1}}$-regular;*

   (b) (Širáň, Tucker and Watkins [66])
   $(4,2,4)_{\begin{pmatrix} l & -m \\ l & m \end{pmatrix}}$ *and* $(4,2,4)_{\begin{pmatrix} l & m \\ m & l \end{pmatrix}}$ *are Δ^{1}-regular;*

   (c) (Coxeter and Moser [33])
   $(4,2,4)_{\begin{pmatrix} l & -m \\ m & l \end{pmatrix}}$ *is Δ^{+}-regular (that is, orientably-regular);*

   (d) (Coxeter and Moser [33])
   $(4,2,4)_{\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}}$ *and* $(4,2,4)_{\begin{pmatrix} k & -k \\ k & k \end{pmatrix}}$ *are Δ-regular (that is, regular).*

2. *(a)* (Coxeter and Moser [33])
   $(6,2,3)_{\begin{pmatrix} l & -l-m \\ m & l \end{pmatrix}}$ *is Δ^{+}-regular (that is, orientably-regular);*

   (b) (Coxeter and Moser [33])
   $(6,2,3)_{\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}}$ *and* $(6,2,3)_{\begin{pmatrix} k & -2k \\ k & k \end{pmatrix}}$ *are Δ-regular (that is, regular).*

Proof. If $U = N\langle X^aY^b, X^cY^d \rangle$ is a hypermap subgroup of a uniform map $\mathcal{U}$ on the torus, then $\mathcal{U}$ is Θ-regular if and only if $U/N \lhd \Theta/N$, that is, if and only if $N(X^aY^b)^g, N(X^cY^d)^g \in U/N$ for every $g \in S$, where S is a set of generators of Θ. This can be easily carried out by using Lemma 4.1.2 and by choosing S as $\{R_1R_2, R_2R_0\}$, $\{R_0, R_2, {R_0}^{R_1}, {R_2}^{R_1}\}$, $\{R_1, R_2R_0, R_0R_1R_2\}$ and $\{R_0, R_1, R_2\}$, according to $\Theta = \Delta^+$, $\Delta^{\hat{1}}$, Δ^1 and Δ, respectively. $\square$

Lemma 4.1.13. *Let* $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ *such that* $\det(M) \neq 0$, (l,m,n) *be* $(4,2,4)$ *or* $(6,2,3)$, $\mathcal{U} = (l,m,n)_M$ *and* $U = \langle (R_1R_2)^l, (R_2R_0)^m, (R_0,R_1)^n \rangle^\Delta \langle X^aY^b, X^cY^d \rangle$ *a hypermap subgroup of* $\mathcal{U}$. *Then the following statements are equivalent:*

1. $X^iY^j \in U$;
2. *there are* $x, y \in \mathbb{Z}$ *such that*
$$\begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{4.1}$$
3. $\det(M) = ad - bc \mid di - cj, aj - bi$.

In addition, if $X^iY^j \in U$, *then* $\gcd(a,c) \mid i$, $\gcd(b,d) \mid j$, $\gcd(a+b, c+d) \mid i+j$ *and* $\gcd(a-b, c-d) \mid i-j$.

Proof. Since $N \subseteq U$,

$$\begin{aligned} X^iY^j \in U \quad & \Leftrightarrow \quad NX^iY^j \in U/N = \langle NX^aY^b, NX^cY^d \rangle \cong \mathbb{Z} \times \mathbb{Z} \\ & \Leftrightarrow \quad \text{there are } x, y \in \mathbb{Z} \text{ such that } (i,j) = x(a,b) + y(c,d) \text{, or equivalently,} \\ & \qquad \begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ & \Leftrightarrow \quad ad - bc \mid (ad-bc)x = di - cj, (ad-bc)y = aj - bi. \end{aligned}$$

Naturally, $\gcd(a,c) \mid ax + cy = i$, $\gcd(b,d) \mid bx + dy = j$, $\gcd(a+b, c+d) \mid (a+b)x + (c+d)y = i + j$ and $\gcd(a-b, c-d) \mid (a-b)x + (c-d)j = i - j$. $\square$

Proposition 4.1.14. *Let* $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ *such that* $\det(M) \neq 0$, $d_1 = \gcd(a,c)$, $d_2 = \gcd(b,d)$, $d^+ = \gcd(a+b, c+d)$ *and* $d^- = \gcd(a-b, c-d)$.

1. *Let* $\mathcal{U} = (4,2,4)_M$, $U = N\langle X^aY^b, X^cY^d \rangle$ *a hypermap subgroup of* $\mathcal{U}$. *Then:*
   (a) $R_0 \in \mathrm{N}_\Delta(U)$ *if and only if* $R_2 \in \mathrm{N}_\Delta(U)$;
   (b) *if* $R_0, R_1 \in \mathrm{N}_\Delta(U)$, *or* $R_2, R_1 \in \mathrm{N}_\Delta(U)$, *then* $\mathrm{N}_\Delta(U) = \Delta$;
   (c) *if* $R_0 \in \mathrm{N}_\Delta(U)$, *or equivalently, if* $R_2 \in \mathrm{N}_\Delta(U)$, *then* $|\det(M)|$ *is* d_1d_2, *or* $|\det(M)|$ *is* $2d_1d_2$ *and* $\frac{a}{d_1} - \frac{b}{d_2}, \frac{c}{d_1} - \frac{d}{d_2}$ *are even;*
   (d) *if* R_1 *is in* $\mathrm{N}_\Delta(U)$, *then* $a - b$ *and* $c - d$ *are even and* $|\det(M)|$ *is* $\frac{d^+d^-}{2}$, *or* $|\det(M)|$ *is* d^+d^- *and* $\frac{a-b}{d^-} - \frac{a+b}{d^+}$ *and* $\frac{c-d}{d^-} - \frac{c+d}{d^+}$ *are even;*
   (e) *if* $R_2R_1 \in \mathrm{N}_\Delta(U)$, *then* $|\det(M)|$ *is* $a^2 + b^2$, $c^2 + d^2$, $(a-b)^2 + (c-d)^2$ *or* $(a+b)^2 + (c+d)^2$, *and* $\det(M)$ *divides* $a^2 + b^2$, $c^2 + d^2$ *and* $ac + bd$;
   (f) $\mathrm{N}_\Delta(U) = \Delta$ *if and only if* $d_1 = d_2$ *and* $|\det(M)|$ *is* ${d_1}^2$, *or* $|\det(M)|$ *is* $2{d_1}^2$ *and* $2d_1$ *divides* d^-.
2. *Let* $\mathcal{U} = (6,2,3)_M$ *and* $U = N\langle X^aY^b, X^cY^d \rangle$ *a hypermap subgroup of* $\mathcal{U}$. *Then:*
   (a) $R_0 \in \mathrm{N}_\Delta(U)$ *if and only if* $R_2 \in \mathrm{N}_\Delta(U)$;
   (b) *if* $R_0, R_1 \in \mathrm{N}_\Delta(U)$ *or* $R_2, R_1 \in \mathrm{N}_\Delta(U)$, *then* $\mathrm{N}_\Delta(U) = \Delta$;

(c) *if* $R_2R_1 \in \mathrm{N}_\Delta(U)$, *then* $|\det(M)|$ *is* a^2+ab+b^2, c^2+cd+d^2, $(a-b)^2+(a-b)(c-d)+(c-d)^2$ *or* $(a+b)^2+(a+b)(c+d)+(c+d)^2$ *and* $\det(M)$ *divides* $a^2+ab+b^2, c^2+cd+d^2, ac+ad+bd$ *(and* $ac+bc+bd=(ac+ad+bd)-(ad-bc)$*);*

(d) $U \lhd \Delta$ *if and only if* $d_1 = d_2$ *and* $|\det(M)|$ *is* ${d_1}^2$ *or* $|\det(M)|$ *is* $3{d_1}^2$ *and* $3d_1$ *divides* d^-.

Proof. 1. (a) By Corollary 4.1.7 $R_2R_0 \in \mathrm{N}_\Delta(U)$.

(b) By (a), $R_0 \in \mathrm{N}_\Delta(U)$ if and only if $R_2 \in \mathrm{N}_\Delta(U)$, so $\mathrm{N}_\Delta(U)$ is a subgroup of Δ containing R_0, R_1, R_2 and hence $\mathrm{N}_\Delta(U) = \Delta$.

(c) By Corollary 4.1.4, $U_\Delta(X^aY^b)^p(X^cY^d)^q = U_\Delta X^{ap+cq}Y^{bp+dq}$. Because X^aY^b and X^cY^d are U, $X^{ap+cq}Y^{bp+dq}$ is also in U. Replacing (p,q) by $(d,-b)$ and by $(-c,a)$, we get $X^{ad-bc}, Y^{ad-bc} \in U$, showing that $|\det(M)| = |ad-bc|$ belongs to the sets $L := \{n \in \mathbb{N} \mid X^n \in U\}$ and $M := \{n \in \mathbb{N} \mid Y^n \in U\}$. Hence, the sets L and M are non-empty. Because $\mathbb{N}$ is a well-ordered set, L and M have minimums. Let $l := \min L$ and $m := \min M$. From the definition of l and m, it follows that if $X^pY^q \in U$, then $l \mid p$ if and only if $m \mid q$. Indeed, if $X^pY^q \in U$, then $l \mid p \Leftrightarrow X^p \in U \Leftrightarrow Y^q \in U \Leftrightarrow m \mid q$.

<u>Case 1</u>: For all $p, q \in \mathbb{Z}$ such that $0 < p < l$, $0 < q < m$, $UX^pY^q \neq U$, that is, $X^pY^q \notin U$. In this case $\mathcal{U}$ can be represented by the Euclidean parallelogram with opposite sides identified and vertices 0, l, mi, $l+mi$. Hence $F = |\det(M)| = lm$. Since $X^l, Y^m \in U$, by Lemma 4.1.13, $d_1 \mid l$, $d_2 \mid m$ and $lm = |ad-bc| \mid dl, -bl, -cm, am$. It follows that $l \mid d_1$ and $m \mid d_2$, and because l, m, d_1, d_2 are non-negative integers, $l = d_1$ and $m = d_2$. In addition there are $r, s, t, u \in \mathbb{Z}$ such that $ru - st = \pm 1$ and

$$\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} r & t \\ s & u \end{pmatrix}. \tag{4.2}$$

<u>Case 2</u>: There are $0 < p < l$ and $0 < q < m$ such that $UX^pY^q = U$, that is, $X^pY^q \in U$. By Corollary 4.1.4, $U_\Delta(X^pY^q)^{R_0} = U_\Delta X^{-p}Y^q$, $U_\Delta(X^pY^q)^{R_2} = U_\Delta X^pY^{-q}$, $U_\Delta X^{2p} = U_\Delta X^pY^q \cdot U_\Delta(X^pY^q)^{R_2}$ and $U_\Delta Y^{2q} = U_\Delta X^pY^q \cdot U_\Delta X^{-p}Y^q$. Because $R_0, R_2 \in \mathrm{N}_\Delta(U)$ and $X^pY^q \in U$, $X^{-p}Y^q$ and X^pY^{-q} are in U, as well as X^{2p} and Y^{2q}. By the definition of l and m (and the Euclidean division algorithm), $l \mid 2p$ and $m \mid 2q$. Furthermore, since l, m, p, q are non-negative integers such that $0 < 2p < 2l$ and $0 < 2q < 2m$, $l = 2p$ and $m = 2q$. In this case $\mathcal{U}$ can be represented by the Euclidean parallelogram with opposite sides identified and vertices $0, p+qi, -p+qi, 2qi$ or $0, p+qi, p-qi, 2p$ (see Figure 4.2). Hence $|\det(M)| = F = lm/2 = 2pq$. Since $X^pY^q, X^{-p}Y^q \in U$, by Lemma 4.1.13, $d_1 \mid p$, $d_2 \mid q$ and $2pq = |ad-bc| \mid dp-cq, dp+cq, bp-aq, bp+aq$. Consequently $2pq \mid 2aq, 2bp, 2cq, 2dp$, so $p \mid d_1$ and $q \mid d_2$. Because p, q, d_1, d_2 are non-negative integers, $p = d_1$ and $q = d_2$. From $2pq \mid bp-aq, dp-cq$, it follows that $\frac{a}{d_1} - \frac{b}{d_2}, \frac{c}{d_1} - \frac{d}{d_2}$ are even. In addition there are $r, s, t, u \in \mathbb{Z}$ such that such that $ru - st = \pm 1$ and

$$\begin{pmatrix} d_1 & -d_1 \\ d_2 & d_2 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} r & t \\ s & u \end{pmatrix}. \tag{4.3}$$

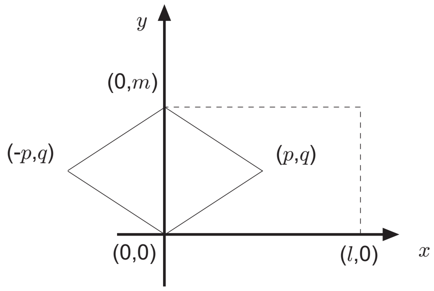

Figure 4.2: $R_0, R_2 \in \mathrm{N}_\Delta(U)$

(d) By Corollary 4.1.4, $U_\Delta X^qY^p = U_\Delta(X^pY^q)^{R_1}$, $U_\Delta X^{p+q}Y^{p+q} = U_\Delta X^pY^q \cdot U_\Delta(X^pY^q)^{R_1}$ and $U_\Delta X^{-(p-q)}Y^{p-q} = U_\Delta(X^pY^q)^{-1} \cdot U_\Delta(X^pY^q)^{R_1}$. If $R_1 \in \mathrm{N}_\Delta(U)$ and $X^pY^q \in U$, then $X^qY^p, X^{p+q}Y^{p+q}, X^{-(p-q)}Y^{p-q} \in U$. Replacing (p,q) by (a,b) and by (c,d), we get $X^{a+b}Y^{a+b}, X^{c+d}Y^{c+d}, X^{-(a-b)}Y^{a-b}, X^{-(c-d)}Y^{c-d} \in U$.

Because $(a+b)d-b(c+d) = (a-b)d-b(c-d) = ad-bc \neq 0$, $a+b$ and $c+d$ cannot be simultaneously 0, as well as $a-b$ and $c-d$. Consequently, $|a+b|$ or $|c+d|$ are in $L := \{n \in \mathbb{N} \mid X^nY^n \in U\}$ and $|a-b|$ or $|c-d|$ are in $M := \{n \in \mathbb{N} \mid X^{-n}Y^n \in U\}$. Hence, the sets L and M are non-empty. Because $\mathbb{N}$ is a well-ordered set, L and M have minimums. Let $l := \min L$ and $m := \min M$. From the definition of l and m, it follows that if $X^pY^q \in U$, then $2l \mid p+q$ if and only if $2m \mid -p+q$. Indeed, if $X^pY^q \in U$, then $2l \mid p+q \Leftrightarrow X^{\frac{p+q}{2}}Y^{\frac{p+q}{2}} \in U \Leftrightarrow X^{-\frac{-p+q}{2}}Y^{\frac{-p+q}{2}} \in U \Leftrightarrow 2m \mid -p+q$.

Case 1: For all p and q such that $0 < p+q < 2l$, $0 < -p+q < 2m$, $UX^pY^q \neq U$, that is, $X^pY^q \notin U$. In this case $\mathcal{U}$ can be represented by the Euclidean parallelogram with opposite sides identified and vertices $0, l+li, -m+mi, (l-m)+(l+m)i$. Hence $F = |\det(M)| = 2lm$. Since $X^lY^l, X^{-m}Y^m \in U$, by Lemma 4.1.13, $d^+ \mid l+l = 2l$, $d^- \mid -m-m = -2m$ and $2lm = |ad-bc| \mid -l(c-d), l(a-b), -m(c+d), m(a+b)$. Thus $2l \mid d^+$ and $2m \mid d^-$. Because l, m, d^+, d^- are non-negative integers, $2l = d^+$ and $2m = d^-$ Naturally, $d^- = 2m$ and $d^+ = 2l$ are even and $|\det(M)| = |ad-bc| = 2lm = \frac{d^+d^-}{2}$. In addition there are $r, s, t, u \in \mathbb{Z}$ such that such that $ru - st = \pm 1$ and

$$\begin{pmatrix} \frac{d^+}{2} & -\frac{d^-}{2} \\ \frac{d^+}{2} & \frac{d^-}{2} \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} r & t \\ s & u \end{pmatrix}. \tag{4.4}$$

Case 2: There are $p, q \in \mathbb{Z}$ such that $0 < p+q < 2l$, $0 < -p+q < 2m$ and $UX^pY^q = U$, that is, $X^pY^q \in U$. Since $X^{p+q}Y^{p+q}, X^{-(p-q)}Y^{p-q} \in U$, by definition of l and m (and the Euclidean division algorithm), $l \mid p+q$ and $m \mid -(p-q)$. Consequently $l = p+q$ and $m = -(p-q)$. In this case $\mathcal{U}$ can be represented by the Euclidean parallelogram with opposite sides identified and vertices $0, p+qi, q+pi, (p+q)+(p+q)i$ (see Figure 4.3). Hence $|\det(M)| = F = 2lm/2 = lm = (p+q)(-p+q)$. By Lemma 4.1.13, $X^pY^q \in U$ implies $d^+ \mid p+q$, $d^- \mid -p+q$, and $(p+q)(-p+q) = |ad-bc| \mid dp-cq, aq-bp, dq-cp, ap-bq$. So $lm = (p+q)(-p+q) \mid -(-p+q)(c+d) = -m(c+d), (p+q)(c-d) = l(c-d), -(-p+q)(a+b) = -m(a+b), (p+q)(a-b) = l(a-b)$. Thus $l \mid d^+$ and $m \mid d^-$. Because l, m, d^+, d^- are non-negative integers, $l = d^+$ and $m = d^-$. From $lm \mid ap-bq, -cq+dp$ we get $2lm \mid 2(ap-bq) = (a-b)l-(a+b)m, 2(-cq+dp) = -(c-d)l-(c+d)m$, so $\frac{a-b}{m} - \frac{a+b}{l}$ and $\frac{c-d}{m} - \frac{c+d}{l}$ must be even. In addition there are $r, s, t, u \in \mathbb{Z}$ such that such that $ru - st = \pm 1$ and

$$\begin{pmatrix} \frac{d^+-d^-}{2} & \frac{d^++d^-}{2} \\ \frac{d^++d^-}{2} & \frac{d^+-d^-}{2} \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} r & t \\ s & u \end{pmatrix}. \tag{4.5}$$

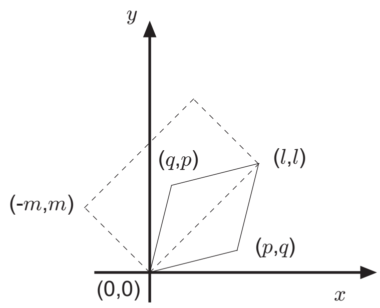

Figure 4.3: $R_1 \in \mathrm{N}_\Delta(U)$

(e) Let $K := \{p^2+q^2 \mid p, q \in \mathbb{Z} \;\&\; (p,q) \neq (0,0) \;\&\; X^pY^q \in U\}$. Since $a^2+b^2, c^2+d^2 \in K$, K is non-empty. Because $\mathbb{N}$ is a well-ordered set, K has a minimum. Let $k := \min K$. Let $l, m \in \mathbb{Z}$ such that $l^2+m^2 = k$. By definition, $l^2+m^2 = k \leq \min\{a^2+b^2, c^2+d^2\}$. Corollary 4.1.4 implies that $U_\Delta(X^pY^q)^{R_2R_1} = U_\Delta X^{-q}Y^p$ and $U_\Delta X^{p-q}Y^{p+q} = U_\Delta X^pY^q \cdot U_\Delta(X^pY^q)^{R_2R_1}$. Since $R_2R_1 \in \mathrm{N}_\Delta(U)$, if $X^pY^q \in U$, then $X^{-q}Y^p, (X^pY^q)^{R_2R_1} \in U$. Replacing (p,q) by $(d,-b)$ and by $(-c,a)$, it follows that $X^{-b}Y^a, X^{-d}Y^c \in U$. By Lemma 4.1.13, $ad - bc \mid a^2+b^2, c^2+d^2, ac+bd$.

We note that $X^0Y^0, X^lY^m, X^{-m}Y^l, X^{l-m}Y^{l+m} \in U$. We now show that $\mathcal{U}$ can be represented by the Euclidean parallelogram with opposite sides identified and with vertices $0, l+mi, -m+li, (l-m)+(l+m)i$ by proving that there are no $p, q \in \mathbb{Z}$ such that $X^pY^q \in U$ and (p,q) is inside of the square with vertices $(0,0)$, (l,m), $(-m,l)$ and $(l-m, l+m)$. Let $p, q \in \mathbb{Z}$ such that and $P := (p,q)$ is inside of the square with vertices $A := (0,0)$, $B := (l,m)$, $C := (-m,l)$ and $D := (l-m, l+m)$, that is, such that $0 < -mp+lq, lp+mq < l^2+m^2 = k$, and let r, s, t, u be the distances of P to A, B, C, D, respectively. Observe that P must lie in at least one of the circles with center in A, B, C, D, and radius $\sqrt{k}$, so the distance of P to each of the 4 vertices of the square cannot be simultaneously greater or equal than $\sqrt{k} = \sqrt{l^2+m^2}$, which is the length of the side of the square (see Figure 4.4). This implies that $X^pY^q \notin U$ because otherwise there would be $Q \in \{A, B, C, D\}$ and $u, v \in \mathbb{Z}$ such that $(u,v) = P - Q$, $X^uY^v \in U$ and $0 < u^2+v^2 < l^2+m^2 = k$, contradicting the minimality of k.

Hence $|ad-bc| = |\det(M)| = F = k = l^2+m^2$. Since $X^lY^m, X^{-m}Y^l \in U$, by Lemma 4.1.13, $l^2+m^2 = |ad-bc|$ divides $al+bm$, $bl-am$, $cl+dm$ and $dl-cm$. By the definition of l and m, (l,m) must be inside of the parallelogram with vertices $\alpha(a,b)+\beta(c,d)$, where $\alpha, \beta \in \{\pm 1\}$ (see figure 4.5); else there would be a lattice-preserving translation τ of the plane such that $(0,0)$ is closer to $(l,m)\tau$ than to (l,m)). Therefore $-(ad-bc) \leq -bl+am, dl-cm \leq ad-bc$. Owing to this, and

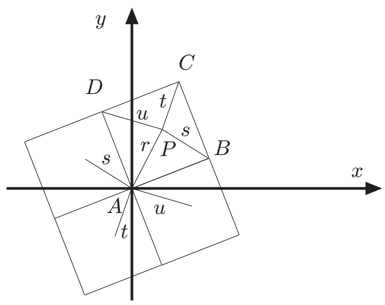

Figure 4.4: r, s, t, u cannot be all greater than $\sqrt{k} = \sqrt{l^2 + m^2}$.

because $|ad - bc| \mid -bl + am, dl - cm$, it follows that $(ld - mc, -lb + ma) = (ku, kv)$, with $u, v \in \{-1, 0, 1\}$.

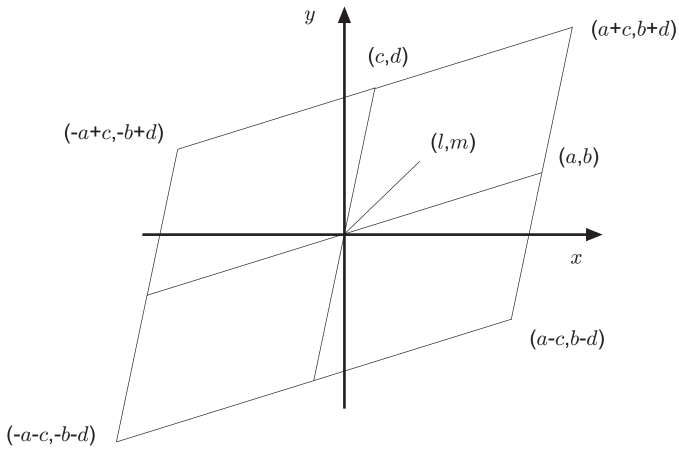

Figure 4.5: (l, m) is inside the parallelogram.

Then:

$$\begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} l \\ m \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

Multiplying both members of this equation by $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$, we get

$$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} l \\ m \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

since $\left(\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}\right)$ comutes with $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$. Hence

$$\begin{pmatrix} l \\ m \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = u \begin{pmatrix} a \\ b \end{pmatrix} + v \begin{pmatrix} c \\ d \end{pmatrix}.$$

This gives the 9 possible values for (l,m), namely $u(a,b)+v(c,d)$, with $u,v \in \{-1,0,1\}$. Since $|\det(M)| > 0$, $|\det(M)| = k = l^2+m^2$ is a^2+b^2, c^2+d^2 or $(a-c)^2+(b-d)^2$, $(a+c)^2+(b+d)^2$. In addition, there are $r,s,t,u \in \mathbb{Z}$ such that such that $ru-st = \pm 1$ and

$$\begin{pmatrix} l & -m \\ m & l \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} r & t \\ s & u \end{pmatrix}. \tag{4.6}$$

(f) Looking back at the proof of (c), if $R_1 \in \mathrm{N}_\Delta(U)$, then $X^m = (Y^{R_1})^m = (Y^m)^{R_1}, Y^l = (X^{R_1})^l = (X^l)^{R_1} \in U^{R_1} = U$, so $m \in A$ and $l \in B$. Hence $l = \min A \le m$ and $m = \min B \le l$. Consequently $d_1 = l = m = d_2$. In addition, taking $d := d_1 = d_2$, there are $r,s,t,u \in \mathbb{Z}$ such that such that $ru-st = \pm 1$ and

$$\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} r & t \\ s & u \end{pmatrix}, \tag{4.7}$$

or

$$\begin{pmatrix} d & -d \\ d & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} r & t \\ s & u \end{pmatrix}. \tag{4.8}$$

The proofs of 2. are similar to those presented in 1. □

We recall that if $g \in \mathrm{N}_\Delta(U)$, then $Ug \in \mathrm{N}_\Delta(U)/U \cong \mathrm{Aut}(\mathcal{U})$ corresponds to the automorphism of $\mathcal{U}$ that maps a flag Ud, with $d \in \Delta$, to Ugd. In particular, if $\mathcal{U}$ is a uniform map on the torus of type $(4,2,4)$, then $R_1 \in \mathrm{N}_\Delta(U)$ if and only if $\mathrm{Aut}(\mathcal{U})$ includes reflections on the diagonals; $R_0, R_2 \in \mathrm{N}_\Delta(U)$ if and only if $\mathrm{Aut}(\mathcal{U})$ includes reflections on vertical or horizontal lines.

Theorem 4.1.15 (Θ-regularity of $(4,2,4)_M$ and $(6,2,3)_M$). *Let $M = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ with $\det(M) \neq 0$, $d_1 = \gcd(a,c)$, $d_2 = \gcd(b,d)$, $d^+ = \gcd(a+b, c+d)$ and $d^- = \gcd(a-b, c-d)$.*

1. Let $\mathcal{U} = (4,2,4)_M$. Then:

(a) $\mathcal{U}$ is $\Delta^{\hat{0}}$-, $\Delta^{\hat{2}}$-, Δ^0-, Δ^2-regular if and only if $a-b$ and $c-d$ are even and $\mathcal{U}$ is regular;

(b) $\mathcal{U}$ is $\Delta^{\hat{1}}$-regular if and only if $|\det(M)|$ is d_1d_2, or $|\det(M)|$ is $2d_1d_2$ and $\frac{a}{d_1} - \frac{b}{d_2}$, $\frac{c}{d_1} - \frac{d}{d_2}$ are even;

(c) $\mathcal{U}$ is Δ^1-regular if and only if $a-b$ and $c-d$ are even and $|\det(M)|$ is $\frac{d^+d^-}{2}$, or $|\det(M)|$ is d^+d^- and $\frac{a-b}{d^-} - \frac{a+b}{d^+}$ and $\frac{c-d}{d^-} - \frac{c+d}{d^+}$ are even;

(d) $\mathcal{U}$ is Δ^+-regular if and only if $|\det(M)|$ is a^2+b^2, c^2+d^2, $(a-b)^2+(c-d)^2$ or $(a+b)^2+(c+d)^2$ and $\det(M)$ divides a^2+b^2, c^2+d^2 and $ac+bd$;

(e) $\mathcal{U}$ is regular if and only if $d_1 = d_2$ and $|\det(M)|$ is ${d_1}^2$, or $|\det(M)|$ is $2{d_1}^2$ and $2d_1$ divides d^-.

2. Let $\mathcal{U} = (6,2,3)_M$. Then:

(a) $\mathcal{U}$ is $\Delta^{\hat{2}}$-, Δ^2-regular if and only if $\mathcal{U}$ is regular;

(b) $\mathcal{U}$ is not $\Delta^{\hat{0}}$-, $\Delta^{\hat{1}}$-, Δ^0-, Δ^1-regular;

(c) $\mathcal{U}$ *is* Δ^+*-regular if and only if* $|\det(M)|$ *is* a^2+ab+b^2*,* c^2+cd+d^2*,* $(a-b)^2+(a-b)(c-d)+(c-d)^2$ *or* $(a+b)^2+(a+b)(c+d)+(c+d)^2$ *and* $\det(M)$ *divides* a^2+ab+b^2*,* c^2+cd+d^2*,* $ac+ad+bd$ *(and* $ac+bc+bd=(ac+ad+bd)-(ad-bc)$*);*

(d) $\mathcal{U}$ *is regular if and only if* $d_1=d_2$ *and* $|\det(M)|$ *is* ${d_1}^2$ *or* $|\det(M)|$ *is* $3{d_1}^2$ *and* $3d_1$ *divides* d^-*.*

Proof. 1. (a) Let $\Theta\in\{\Delta^{\hat{0}},\Delta^{\hat{2}},\Delta^0,\Delta^2\}$. By Proposition 4.1.14, if $\Theta\subseteq \mathrm{N}_\Delta(U)$, then $\mathrm{N}_\Delta(U)=\Delta$. Hence $\mathcal{U}$ is Θ-regular if and only if $\mathcal{U}$ is Θ-conservative, that is, if $a-b$ and $c-d$ are even (Proposition 4.1.8).

(b), (c), (d), (e) Let $\Theta\in\{\Delta^{\hat{1}},\Delta^1,\Delta^+,\Delta\}$. By Proposition 4.1.8 every uniform map on the torus of type $(4,2,4)$ is Θ-conservative.
($\Rightarrow$)'s are consequences of 1.(c), 1.(d), 1.(e), 1.(f) of Proposition 4.1.14, respectively;
($\Leftarrow$)'s Using (4.2), (4.3), (4.4), (4.5), (4.6), (4.7) and (4.8), together with Theorem 4.1.10, we can see that $\mathcal{U}$ is isomorphic to one of the hypermaps listed in 1. of Proposition 4.1.12; by this result $\mathcal{U}$ is $\Delta^{\hat{1}}$-, Δ^1-, Δ^+-, or Δ-regular, respectively.

2. (a) Let $\Theta\in\{\Delta^{\hat{2}},\Delta^2\}$. Every uniform map on the torus of type $(6,2,3)$ is Θ-conservative by Proposition 4.1.8. By 2.(b) of Proposition 4.1.14, $\mathcal{U}$ is Θ-regular if and only if $\mathcal{U}$ is regular.

(b) If $\Theta\in\{\Delta^{\hat{0}},\Delta^{\hat{1}},\Delta^0,\Delta^1\}$, then $\mathcal{U}$ is not Θ-conservative (Proposition 4.1.8), hence $\mathcal{U}$ is not Θ-regular.

(c), (d) Let $\Theta\in\{\Delta^+,\Delta\}$. As in 1.(d), 1. (f), $\mathcal{U}$ is always Θ-conservative.
($\Rightarrow$)'s are consequences of 2.(c), 2.(d) of Proposition 4.1.14;
($\Leftarrow$)'s follow from 2. of Lemma 4.1.13.

□

Corollary 4.1.16. *Let* $M=\begin{pmatrix} a & c\\ b & d\end{pmatrix}$ *such that* $\det(M)\neq 0$*.*

1. *(a)* $(4,2,4)_M$ *is* $\Delta^{\hat{1}}$*-regular if and only if* $(4,2,4)_M\cong(4,2,4)_N$*, where* N *is* $\begin{pmatrix} l & 0\\ 0 & m\end{pmatrix}$ *or* $\begin{pmatrix} l & -l\\ m & m\end{pmatrix}$*, for some* $l,m\in\mathbb{N}$*;*

   (b) (Širáň, Tucker and Watkins [66])
   $(4,2,4)_M$ *is* Δ^1*-regular if and only if* $(4,2,4)_M\cong(4,2,4)_N$*, where* N *is* $\begin{pmatrix} l & -m\\ l & m\end{pmatrix}$ *or* $\begin{pmatrix} l & m\\ m & l\end{pmatrix}$*, for some* $l,m\in\mathbb{N}$*;*

   (c) (Coxeter and Moser [33])
   $(4,2,4)_M$ *is* Δ^+*-regular if and only if* $(4,2,4)_M\cong(4,2,4)_N$*, where* N *is* $\begin{pmatrix} l & -m\\ m & l\end{pmatrix}$*, for some* $l,m\in\mathbb{N}$*;*

   (d) (Coxeter and Moser [33])
   $(4,2,4)_M$ *is regular if and only if* $(4,2,4)_M\cong(4,2,4)_N$*, where* N *is* $\begin{pmatrix} k & 0\\ 0 & k\end{pmatrix}$ *or* $\begin{pmatrix} k & -k\\ k & k\end{pmatrix}$*, for some* $k\in\mathbb{N}$*.*

2. *(a)* (Coxeter and Moser [33])
   $(6,2,3)_M$ *is* Δ^+*-regular if and only if* $(6,2,3)_M\cong(6,2,3)_N$*, where* N *is* $\begin{pmatrix} l & -l-m\\ m & l\end{pmatrix}$*, for some* $l,m\in\mathbb{N}$*;*

(b) (Coxeter and Moser [33])
$(6,2,3)_M$ *is regular if and only if* $(6,2,3)_M \cong (6,2,3)_N$, *where* N *is* $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ *or* $\begin{pmatrix} k & -2k \\ k & k \end{pmatrix}$, *for some* $k \in \mathbb{N}$.

Note that the orientably-regular hypermaps $(4,2,4)_{\begin{pmatrix} l & -m \\ m & l \end{pmatrix}}$ and $(6,2,3)_{\begin{pmatrix} l & -l-m \\ m & l \end{pmatrix}}$ ($\{4,4\}_{l,m}$ and $\{3,6\}_{l,m}$ in the notation of Coxeter and Moser [33]) are regular if and only if $l = 0$ or $m = 0$ or $l = m$.

Remark 4.1.17. In [66], Širáň, Tucker and Watkins proved that a uniform map $\mathcal{U}$ of type $(4,2,4)$ is Δ^1-regular if and only if $\mathcal{U} \cong (4,2,4)_{\begin{pmatrix} r_1 & -s \\ r_2 & s \end{pmatrix}}$, where $s \mid r_1 - r_2$. If $(r_1 - r_2)/s$ is even, say $r_1 - r_2 = 2ks$, then

$$\begin{pmatrix} r_1 & -s \\ r_2 & s \end{pmatrix} = \begin{pmatrix} r_2 + 2ks & -s \\ r_2 & s \end{pmatrix} = \begin{pmatrix} r_2 + ks & -s \\ r_2 + ks & s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix}$$

and $(4,2,4)_{\begin{pmatrix} r_1 & -s \\ r_2 & s \end{pmatrix}} \cong (4,2,4)_{\begin{pmatrix} r_2+ks & -s \\ r_2+ks & s \end{pmatrix}}$ since $\det \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} = 1$; else if $(r_1 - r_2)/s$ is odd, say $r_1 - r_2 = (2k+1)s$, then

$$\begin{aligned} \begin{pmatrix} r_1 & -s \\ r_2 & s \end{pmatrix} &= \begin{pmatrix} r_2 + s + 2ks & -s \\ r_2 & s \end{pmatrix} = \begin{pmatrix} r_2 + s + ks & -s \\ r_2 + ks & s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} \\ &= \begin{pmatrix} r_2 + s + ks & r_2 + ks \\ r_2 + ks & r_2 + s + ks \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} \\ &= \begin{pmatrix} r_2 + s + ks & r_2 + ks \\ r_2 + ks & r_2 + s + ks \end{pmatrix} \begin{pmatrix} 1+k & -1 \\ -k & 1 \end{pmatrix} \end{aligned}$$

and $(4,2,4)_{\begin{pmatrix} r_1 & -s \\ r_2 & s \end{pmatrix}} \cong (4,2,4)_{\begin{pmatrix} r_2+s+ks & r_2+ks \\ r_2+ks & r_2+s+ks \end{pmatrix}}$ since $\det \begin{pmatrix} 1+k & -1 \\ -k & 1 \end{pmatrix} = 1$.

It is also shown that a uniform map $\mathcal{U}$ of type $(6,2,3)$ is orientably-regular if and only if $\mathcal{U} \cong (6,2,3)_{\begin{pmatrix} r & -s \\ s & r+s \end{pmatrix}}$. However, since

$$\begin{pmatrix} r & -s \\ s & r+s \end{pmatrix} = \begin{pmatrix} r & -r-s \\ s & r \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and $\det \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = 1$, $(6,2,3)_{\begin{pmatrix} r & -s \\ s & r+s \end{pmatrix}} \cong (6,2,3)_{\begin{pmatrix} r & -r-s \\ s & r \end{pmatrix}}$.

Remark 4.1.18. One can easily see that

$$\begin{pmatrix} 2k & k \\ 0 & k \end{pmatrix} = \begin{pmatrix} k & -k \\ k & k \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \ \begin{pmatrix} 3k & k \\ 0 & k \end{pmatrix} = \begin{pmatrix} k & -2k \\ k & k \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \det \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = 1.$$

Let $\mathcal{R} = (l,m,n)_M$, where (l,m,n) is $(4,2,4)$ or $(6,2,3)$ and M is $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ or $\begin{pmatrix} kl/2 & k \\ 0 & k \end{pmatrix}$, for some $k \in \mathbb{N}$. Then $\mathcal{R}$ is a regular map and the automorphism group, $\mathrm{Aut}(\mathcal{R})$, and the rotation group, $\mathrm{Aut}^+(\mathcal{R})$, of $\mathcal{R}$ are isomorphic to Δ/R and Δ^+/R, respectively. Since $\Delta/R = \langle RX, RY, RR_1, RR_2 \rangle$, $\Delta^+/R = \langle RX, RY, RR_1R_2 \rangle$, $\langle RX, RY \rangle$ is a normal subgroup of Δ/R (see Corollary 4.1.4) and $\langle RX, RY \rangle \cap \langle RR_1, RR_2 \rangle = \{1\}$, we have

$$\mathrm{Aut}(\mathcal{R}) \cong (C_k \times C_k) \rtimes D_l \text{ and } \mathrm{Aut}^+(\mathcal{R}) \cong (C_k \times C_k) \rtimes C_l, \text{ if } M \text{ is } \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, \tag{4.9}$$

and

$$\mathrm{Aut}(\mathcal{R}) \cong (C_{kl/2} \times C_k) \rtimes D_l \text{ and } \mathrm{Aut}^+(\mathcal{R}) \cong (C_{kl/2} \times C_k) \rtimes C_l, \text{ if } M \text{ is } \begin{pmatrix} kl/2 & k \\ 0 & k \end{pmatrix}. \quad (4.10)$$

As remarked by Coxeter and Moser (see Table 7 of [33]), the automorphism groups of $(4,2,4)_{\left(\begin{smallmatrix} k & -k \\ k & k \end{smallmatrix}\right)}$ and $(6,2,3)_{\left(\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}\right)}$ are the groups $G^{4,4,2k}$ and $G^{3,6,2k}$, with orders $16k^2$ and $12k^2$ respectively.

4.1.2 Uniform hypermaps on the torus of type $(3,3,3)$

The uniform hypermaps on the torus of type $(3,3,3)$ can be obtained from the uniform maps on the torus of type $(6,2,3)$ in the following way. If $\mathcal{U}$ is a uniform hypermap on the torus of type $(3,3,3)$, then Walsh$(\mathcal{U})$ is a bipartite-uniform hypermap on the torus of bipartite-type $(3,3;2;6)$. Owing to this, Walsh$(\mathcal{U})$ and $\mathcal{M} := \mathrm{D}_{(02)}(\mathrm{Walsh}(\mathcal{U}))$ are uniform maps on the torus of types $(3,2,6)$ and $(6,2,3)$. Reciprocally, if $\mathcal{M}$ is a uniform map on the torus of type $(6,2,3)$, then, by Proposition 4.1.8, $\mathcal{M}$ is $\Delta^{\hat{2}}$-conservative and $\mathrm{D}_{(02)}(\mathcal{M})$ is $\Delta^{\hat{0}}$-conservative, that is, bipartite. Now, $\mathrm{D}_{(02)}(\mathcal{M})$, being bipartite and uniform, is bipartite-uniform and its bipartite-type is $(3,3;2;6)$. Then, by Theorem 1.6.5, $\mathrm{D}_{(02)}(\mathcal{M}) \cong \mathrm{Walsh}(\mathcal{U})$, for some uniform hypermap $\mathcal{U}$ on the torus of type $(3,3,3)$. Furthermore, if $\mathcal{V}$ is another hypermap such that $\mathrm{Walsh}(\mathcal{V}) \cong \mathrm{D}_{(02)}(\mathcal{M}) \cong \mathrm{Walsh}(\mathcal{U})$, then, by Theorem 1.6.6, $\mathcal{V} \cong \mathcal{U}$ or $\mathcal{V} \cong \mathrm{D}_{(01)}(\mathcal{U})$. If U and M are hypermap subgroups of $\mathcal{U}$ and $\mathcal{M}$, then $M = U{\varphi_W}^{-1}\overline{(02)}$, $U{\varphi_W}^{-1} = M\overline{(02)}$ and $U = U{\varphi_W}^{-1}\varphi_W = M\overline{(02)}\varphi_W$.

Let $X_3 := X_6\overline{(02)}\varphi_W = R_2R_1R_0R_1 = {Y_4}^{-1}$, $Y_3 := Y_6\overline{(02)}\varphi_W = R_1R_2R_1R_0 = {X_4}^{-1}$ and

$$\begin{aligned} N_3 &:= N_6\overline{(02)}\varphi_W \\ &= \langle (R_1R_2)^6, (R_2R_0)^2, (R_0R_1)^3 \rangle^{\Delta}\overline{(02)}\varphi_W \\ &= \langle (R_1R_2)^3, (R_2R_0)^2, (R_0R_1)^6 \rangle^{\Delta}\varphi_W \\ &= \langle (R_1R_2)^3, [(R_1R_2)^3]^{R_0}, (R_2R_0)^2, (R_0R_1)^6 \rangle^{\Delta^{\hat{0}}}\varphi_W \\ &= \langle (R_1R_2)^3, ({R_1}^{R_0}{R_2}^{R_0})^3, R_2{R_2}^{R_0}, ({R_1}^{R_0}R_1)^3 \rangle^{\Delta^{\hat{0}}}\varphi_W \\ &= \langle (R_1R_2)^3, (R_2R_0)^3, (R_0R_1)^3 \rangle^{\Delta}. \end{aligned}$$

Clearly, $Y_3 = {X_4}^{-1} = ({Y_4}^{-1})^{R_1} = {X_3}^{R_1}$. Let $\varphi_W^* : \Delta^{\hat{2}} \to \Delta$, $g \mapsto g\overline{(02)}\varphi_W$. Since $N_6\varphi_W^* = N_3$, φ_W^* induces an epimorphism $\Phi_W^* : \Delta^{\hat{2}}/N_6 \to \Delta/N_3$, such that $(N_6g)\Phi_W^* = N_3(g\varphi_W^*)$. By abuse of language, we speak of φ_W^*, meaning Φ_W^*.

Lemma 4.1.19 (Properties of N_3, X_3 and Y_3)**.**

1. *$N_3X_3 \leftrightharpoons N_3Y_3$;*
2. *$N_3{X_3}^{R_0} = N_3X_3{Y_3}^{-1}$, $N_3{X_3}^{R_1} = N_3Y_3$, $N_3{X_3}^{R_2} = N_3{X_3}^{-1}$,*
   $N_3{Y_3}^{R_0} = N_3{Y_3}^{-1}$, $N_3{Y_3}^{R_1} = N_3X_3$, $N_3{Y_3}^{R_2} = N_3{X_3}^{-1}Y_3$.

Proof. This Lemma follows from the definitions of N_3, X_3 and Y_3, the fact that φ_W^* is a group epimorphism and 2. of Lemma 4.1.2.
1. $N_3X_3N_3Y_3 = N_3X_3Y_3 = (N_6X_6Y_6)\varphi_W^* = (N_6Y_6X_6)\varphi_W^* = N_3Y_3X_3 = N_3Y_3N_3X_3$.
2. Since $R_0 = {R_1}^{R_2}\varphi_W^*$, $R_1 = R_1\varphi_W^*$ and $R_2 = R_0\varphi_W^*$, we have:

$N_3X_3{}^{R_0} = (N_6X_6{}^{R_2R_1R_2})\varphi^*_W = (N_6X_6Y_6{}^{-1})\varphi^*_W = N_3X_3Y_3{}^{-1}$,
$N_3X_3{}^{R_1} = (N_6X_6{}^{R_1})\varphi^*_W = (N_6Y_6)\varphi^*_W = N_3Y_3$,
$N_3X_3{}^{R_2} = (N_6X_6{}^{R_0})\varphi^*_W = (N_6X_6{}^{-1})\varphi^*_W = N_3X_3{}^{-1}$,
$N_3Y_3{}^{R_0} = (N_6Y_6{}^{R_2R_1R_2})\varphi^*_W = (N_6Y_6{}^{-1})\varphi^*_W = N_3Y_3{}^{-1}$,
$N_3Y_3{}^{R_1} = (N_6Y_6{}^{R_1})\varphi^*_W = (N_6X_6)\varphi^*_W = N_3X_3$,
$N_3Y_3{}^{R_2} = (N_6Y_6{}^{R_0})\varphi^*_W = (N_6X_6{}^{-1}Y_6)\varphi^*_W = N_3X_3{}^{-1}Y_3$. □

We shall omit the index 3 in N_3, X_3 and Y_3 if it is clear from the context.

Because N is a normal subgroup of Δ contained in U, $N \subseteq U_\Delta$ and hence:

Corollary 4.1.20. *Let $\mathcal{U}$ be a uniform hypermap on the torus of type $(3,3,3)$ and U a hypermap subgroup of $\mathcal{U}$. Then:*

1. $U_\Delta X \leftrightharpoons U_\Delta Y$;
2. $U_\Delta X^{R_0} = U_\Delta XY^{-1}$, $U_\Delta X^{R_1} = U_\Delta Y$, $U_\Delta X^{R_2} = U_\Delta X^{-1}$,
   $U_\Delta Y^{R_0} = U_\Delta Y^{-1}$, $U_\Delta Y^{R_1} = U_\Delta X$, $U_\Delta Y^{R_2} = U_\Delta X^{-1}Y$.

Remark 4.1.21.
1. $N_3\overline{(01)} = N_3$;
   Therefore, $\overline{(01)}$ induces an isomorphism $\Delta/N_3 \to \Delta/N_3$, $N_3g \mapsto N_3g\overline{(01)}$, which, by abuse of language, we also denote by $\overline{(01)}$;
2. $N_3X_3\overline{(01)} = N_3R_2R_0R_1R_0 = N_3X_3(R_1R_0)^3 = N_3X_3 = (N_3X_3{}^{R_2})^{-1}$;
3. $N_3Y_3\overline{(01)} = (N_3X_3{}^{R_1})\overline{(01)} = (N_3X_3\overline{(01)})^{R_1\overline{(01)}} = N_3X_3{}^{R_0} = N_3X_3Y_3{}^{-1} = (N_3Y_3{}^{R_2})^{-1}$;

Lemma 4.1.22. *If $\mathcal{U}$ is a uniform hypermap on the torus of type $(3,3,3)$, then $\mathcal{U} \cong \mathrm{D}_{(01)}(\mathcal{U})$.*

Proof. Let $\mathcal{V} = \mathrm{D}_{(01)}(\mathcal{U})$ and $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ such that $(6,2,3)_M \cong \mathrm{D}_{(02)}(\mathrm{Walsh}(\mathcal{U}))$. Then $U = (N_6\langle X_6{}^aY_6{}^b, X_6{}^cY_6{}^d\rangle)\varphi^*_W = N_3\langle X_3{}^aY_3{}^b, X_3{}^cY_3{}^d\rangle$ and $V = U\overline{(01)}$ are hypermap subgroups of $\mathcal{U}$ and $\mathcal{V}$. By Remark 4.1.21 $N_3X_3\overline{(01)} = (N_3X_3{}^{R_2})^{-1}$ and $N_3Y_3\overline{(01)} = (N_3Y_3{}^{R_2})^{-1}$, so

$$\begin{aligned}
V/N_3 &= U\overline{(01)}/N_3 \\
&= (N_3\langle X_3{}^aY_3{}^b, X_3{}^cY_3{}^d\rangle)\overline{(01)}/N_3 \\
&= (\langle N_3[(X_3{}^aY_3{}^b)\overline{(01)}], N_3[(X_3{}^cY_3{}^d)\overline{(01)}]\rangle) \\
&= \langle [N_3(X_3{}^aY_3{}^b)^{R_2}]^{-1}, [N_3(X_3{}^cY_3{}^d)^{R_2}]^{-1}\rangle \\
&= \langle N_3(X_3{}^aY_3{}^b)^{R_2}, N_3(X_3{}^cY_3{}^d)^{R_2})\rangle \\
&= \langle N_3X_3{}^aY_3{}^b, N_3X_3{}^cY_3{}^d\rangle^{R_2} \\
&= U^{R_2}/N_3.
\end{aligned}$$

Thus $V = U\overline{(01)} = U^{R_2}$. Having conjugate hypermap subgroups, $\mathcal{U}$ and $\mathcal{V}$ are isomorphic. □

We denote by $(3,3,3)_M$ the uniform hypermap on the torus of type $(3,3,3)$ (unique up to isomorphism) such that $\mathrm{Walsh}((3,3,3)_M) \cong \mathrm{D}_{(02)}((6,2,3)_M)$.

Lemma 4.1.23. *The hypermap $\mathcal{U} = (3,3,3)_M$ has $|\Omega_{\mathcal{U}}| = 6|\det(M)|$ flags, $V = |\det(M)|$ vertices, $E = |\det(M)|$ edges and $F = |\det(M)|$ faces.*

Proof. The number of flags of $(3,3,3)_M$ is half the number of flags of $\mathrm{D}_{(02)}((6,2,3)_M)$, which is $12|\det(M)|$. The numbers of vertices, edges and faces of $\mathcal{U}$ are given by the formula $|\Omega_{\mathcal{U}}| = 2lV = 2mE = 2nF$. □

Theorem 4.1.24 (Hypermap subgroup of $(3,3,3)_M$)**.** *The hypermap $\mathcal{U} = (3,3,3)_M$ has hypermap subgroup $U = N\langle X^aY^b, X^cY^d\rangle = \langle (R_1R_2)^6, (R_2R_0)^2, (R_0,R_1)^3\rangle^\Delta \langle X^aY^b, X^cY^d\rangle$.*

Proposition 4.1.25 (Θ-conservativeness of $(3,3,3)_M$)**.** *The uniform hypermap $(3,3,3)_M$ is Δ^+-conservative but is not Θ-conservative for any other $\Theta \lhd_2 \Delta$.*

Proof. Since $(R_1R_2)^3, (R_2R_0)^3, (R_0R_1)^3, X_3 = R_2R_1R_0R_1, Y_3 = R_1R_2R_1R_0 \in \Delta^+$, $U \subseteq \Delta^+$. However, $(R_iR_j)^3 \notin \Delta^{\hat{k}}, \Delta^k$ for every $k \in \{0,1,2\}$, so $U \nsubseteq \Theta$, for $\Theta \lhd_2 \Delta$, $\Theta \neq \Delta^+$. □

Lemma 4.1.26. $(3,3,3)_M \to (3,3,3)_{M'}$ *if and only if* $(6,2,3)_M \to (6,2,3)_{M'}$.

Proof. Let U and U' be hypermap subgroups of $(3,3,3)_M$ and $(3,3,3)_{M'}$. Then $V := U{\varphi^*_W}^{-1}$ and $V' := U'{\varphi^*_W}^{-1}$ are hypermap subgroups of $(6,2,3)_M$ $(6,2,3)_{M'}$. If $(3,3,3)_M \to (3,3,3)_{M'}$, then $U \subseteq (U')^y$, for some $y \in \Delta$. Since φ^*_W is an epimorphism, there is $x \in \Delta^{\hat{2}}$ such that $x\varphi^*_W = y$. Hence $V = U{\varphi^*_W}^{-1} \subseteq (U')^y{\varphi^*_W}^{-1} = (U')^{x\varphi^*_W}{\varphi^*_W}^{-1} = (U'{\varphi^*_W}^{-1})^x = (V')^x$, that is, $(6,2,3)_M \to (6,2,3)_{M'}$.

Reciprocally, if $(6,2,3)_M \to (6,2,3)_{M'}$, then there is $g \in \Delta$ such that $U{\varphi^*_W}^{-1} \subseteq (U'{\varphi^*_W}^{-1})^g$. If $g \in \Delta^{\hat{2}}$, then $U = V\varphi^*_W \subseteq (V')^g\varphi^*_W = (V'\varphi^*_W)^{g\varphi^*_W} = (U')^{g\varphi^*_W}$; else, if $g \notin \Delta^{\hat{2}}$, then $R_2R_0g \in \Delta^{\hat{2}}$ and, by Lemma 4.1.7, $(V'\overline{(02)})^{R_2R_0} = V'\overline{(02)}$, or equivalently, $(V')^{R_0R_2} = V'$, so $U = V\varphi^*_W \subseteq (V')^g\varphi^*_W = [(V')^{R_0R_2}]^{R_2R_0g}\varphi^*_W = (V')^{R_2R_0g}\varphi^*_W = (V'\varphi^*_W)^{R_2R_0g\varphi^*_W} = (U')^{R_0R_2g\varphi^*_W}$. □

Using the previous Lemma together with Theorem 4.1.10 we get:

Theorem 4.1.27.

1. $\mathcal{U} \to \mathcal{U}'$ *if and only if there are* $P \in \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \right\rangle \cong D_6$ *and* $Q \in M(2,\mathbb{Z})$ *such that* $\det(Q) \neq 0$ *and* $M = PM'Q$.

2. $\mathcal{U} \cong \mathcal{U}'$ *if and only if there are* $P \in \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \right\rangle \cong D_6$ *and* $Q \in GL(2,\mathbb{Z})$ *such that* $M = PM'Q$.

Now we give examples of restrictedly-regular uniform hypermaps on the torus of type $(3,3,3)$. The proof is similar to the proof of Proposition 4.1.12.

Proposition 4.1.28. *Let* $k, l, m \in \mathbb{Z}$.

1. (Corn and Singerman [28])
$(3,3,3)_{\begin{pmatrix} l & -l-m \\ m & l \end{pmatrix}}$ *is Δ^+-regular, that is, orientably-regular;*

2. (Corn and Singerman [28] together with Breda and Nedela [10][1])
$(3,3,3)_{\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}}$ *and* $(3,3,3)_{\begin{pmatrix} k & -2k \\ k & k \end{pmatrix}}$ *are Δ-regular, that is, regular.*

[1]Theorem 11 of [28] states that a uniform hypermap $\mathcal{H}$ on the torus of type $(3,3,3)$ is orientably-regular if and only if Walsh($\mathcal{H}$) is orientably-regular. By Theorem 1 of [10], $\mathcal{H}$ is orientably-chiral if and only if Walsh($\mathcal{H}$) is orientably-chiral, and hence $\mathcal{H}$ is regular if and only if Walsh($\mathcal{H}$) is regular.

Finally we have:

Theorem 4.1.29 (Θ-regularity of $(3,3,3)_M$). *Let* $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ *such that* $\det(M) \neq 0$.

1. $(3,3,3)_M$ *is* Δ^+*-regular if and only if* $(6,2,3)_M$ *is* Δ^+*-regular, that is, if and only if* $|\det(M)|$ *is* a^2+ab+b^2, c^2+cd+d^2, $(a-b)^2+(a-b)(c-d)+(c-d)^2$ *or* $(a+b)^2+(a+b)(c+d)+(c+d)^2$, *and* $\det(M)$ *divide* a^2+ab+b^2, c^2+cd+d^2, $ac+ad+bd$ *(and* $ac+bc+bd = (ac+ad+bd)-(ad-bc)$*);*

2. $(3,3,3)_M$ *is not* $\Delta^{\hat{0}}$-, $\Delta^{\hat{1}}$-, $\Delta^{\hat{2}}$-, Δ^{0}-, Δ^{1}-, Δ^{2}*-regular;*

3. $(3,3,3)_M$ *is regular if and only if* $(6,2,3)_M$ *is regular, that is, if and only if* $d_1 = d_2$ *and* $|\det(M)| = {d_1}^2$ *or* $|\det(M)| = 3{d_1}^2$ *and* $3d_1 \mid \gcd(a-b, c-d)$*, where* $d_1 = \gcd(a,c)$ *and* $d_2 = \gcd(b,d)$*.*

Proof. If $\Theta \lhd_2 \Delta$, $\Theta \neq \Delta^+$, then $(3,3,3)_M$ is not Θ-regular, because $(3,3,3)_M$ is not Θ-conservative (Proposition 4.1.25).

Now let U and V be hypermap subgroups of $\mathcal{U} := (3,3,3)_M$ and $\mathcal{V} := (6,2,3)_M$. Then $V = U{\varphi_W}^{-1}\overline{(02)}$ and $U \lhd \Delta \Leftrightarrow U{\varphi_W}^{-1} \lhd \Delta{\varphi_W}^{-1} = \Delta^{\hat{0}} \Leftrightarrow V \lhd \Delta^{\hat{2}} \Leftrightarrow V \lhd \Delta$ (see Theorem 4.1.15). Similarly $U \lhd \Delta^+ \Leftrightarrow U{\varphi_W}^{-1} \lhd \Delta^+{\varphi_W}^{-1} = \Delta^+ \cap \Delta^{\hat{0}} \Leftrightarrow V \lhd \Delta^+ \cap \Delta^{\hat{2}}$. Owing to this, and because $V^{R_2R_0} = V$ (see Corollary 4.1.7), $\Delta^+ = \langle \Delta^+ \cap \Delta^{\hat{2}}, R_2R_0\rangle \subseteq \mathrm{N}_\Delta(V)$, that is $V \lhd \Delta$. $\square$

Using this Theorem, together with

Corollary 4.1.30. *Let* $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ *such that* $\det(M) \neq 0$.

1. (Corn and Singerman [28])
$(3,3,3)_M$ *is* Δ^+*-regular if and only if* $(3,3,3)_M \cong (3,3,3)_{\begin{pmatrix} l & -l-m \\ m & l \end{pmatrix}}$ *for some* $l, m \in \mathbb{N}$*;*

2. (Corn and Singerman [28])
$(3,3,3)_M$ *is regular if and only if* $(3,3,3)_M \cong (3,3,3)_{\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}}$ *or* $(3,3,3)_M \cong (3,3,3)_{\begin{pmatrix} k & -2k \\ k & k \end{pmatrix}}$*, for some* $k \in \mathbb{N}$*.*

Remark 4.1.31. Let $\mathcal{R} = (3,3,3)_M$, where M is $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ or $\begin{pmatrix} k & -2k \\ k & k \end{pmatrix}$, for some $k \in \mathbb{N}$. As in Remark 4.1.18, $\mathcal{R}$ is a regular map and the automorphism group, $\mathrm{Aut}(\mathcal{R})$, and the rotation group, $\mathrm{Aut}^+(\mathcal{R})$, of $\mathcal{R}$ are isomorphic to Δ/R and Δ^+/R, respectively. Since $\Delta/R = \langle RX, RY, RR_1, RR_2\rangle$, $\Delta^+/R = \langle RX, RY, RR_1R_2\rangle$, $\langle RX, RY\rangle$ is a normal subgroup of Δ/R (see Corollary 4.1.20) and $\langle RX, RY\rangle \cap \langle RR_1, RR_2\rangle = \{1\}$, we have

$$\mathrm{Aut}(\mathcal{R}) \cong (C_k \times C_k) \rtimes D_3 \text{ and } \mathrm{Aut}^+(\mathcal{R}) \cong (C_k \times C_k) \rtimes C_3, \text{ if } M \text{ is } \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, \tag{4.11}$$

and

$$\mathrm{Aut}(\mathcal{R}) \cong (C_{3k} \times C_k) \rtimes D_3 \text{ and } \mathrm{Aut}^+(\mathcal{R}) \cong (C_{3k} \times C_k) \rtimes C_3, \text{ if } M \text{ is } \begin{pmatrix} k & -2k \\ k & k \end{pmatrix}. \tag{4.12}$$

4.2 Bipartite-uniform hypermaps on the torus

Let $\mathcal{B}$ is a bipartite-uniform hypermap on the torus of bipartite-type $(l_1, l_2; m; n)$. As before, we may assume, without loss of generality, that $l_1 \leq l_2$ and $m \leq n$. Then, by Lemma 1.3.6, m and n are even. Replacing $\chi_{\mathcal{B}} = 0$ in the Euler formula for bipartite-uniform hypermaps (Corollary 1.4.3), it follows from Lemma 1.4.7 that $l_1 = 1$ or $m/2 = 1$ or $l_1 = l_2 = m/2 = n/2 = 2$. When $l_1 = 1$ or $m/2 = 1$, Theorems 1.6.5 and 1.6.9 imply that $\mathcal{B} \cong \mathrm{Pin}(\mathcal{U})$ or $\mathcal{B} \cong \mathrm{Walsh}(\mathcal{U})$, for some uniform hypermap $\mathcal{U}$ on the torus; in addition, $\mathcal{B}$ is bipartite-regular if and only if $\mathcal{U}$ is regular. When $l_1 = l_2 = m/2 = n/2 = 2$, $\mathcal{B}$ is uniform of type $(2,4,4)$ and so $\mathcal{B} \cong \mathrm{D}_{(01)}((4,2,4)_M)$, for some $M \in M(2,\mathbb{Z})$ such that $\det(M) \neq 0$. Obviously, $\mathcal{B}$ is bipartite-regular if and only if $\mathrm{D}_{(01)}(\mathcal{B}) \cong (4,2,4)_M$ is $\Delta^{\hat{1}}$-regular.

Theorem 4.2.1. *If $\mathcal{B}$ is a bipartite-uniform hypermap on the torus, then $\mathcal{B} \cong \mathrm{Walsh}(\mathcal{U})$ or $\mathcal{B} \cong \mathrm{Pin}(\mathcal{U})$, for some uniform hypermap $\mathcal{U}$ on the torus, or $\mathrm{D}_{(01)}(\mathcal{B})$ is a uniform map on the torus of type $(4,2,4)$. Furthermore, $\mathcal{B}$ is bipartite regular if and only if $\mathcal{B} \cong \mathrm{Walsh}(\mathcal{R})$ or $\mathcal{B} \cong \mathrm{Pin}(\mathcal{R})$ for some regular hypermap $\mathcal{R}$ on the torus, or if $\mathrm{D}_{(01)}(\mathcal{B}) \cong (4,2,4)_M$, where M is $\begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix}$ or $\begin{pmatrix} l & -l \\ m & m \end{pmatrix}$, for some $l, m \in \mathbb{N}$.*

#	l_1	l_2	m	n	V_1	V_2	E	F	$\lvert\Omega\rvert$	$\mathcal{B}$
1	1	2	6	12	$6k$	$3k$	$2k$	k	$24k$	$\mathrm{Pin}(\mathrm{D}_{(021)}((6,2,3)_M))$
2	1	2	8	8	$4k$	$2k$	k	k	$16k$	$\mathrm{Pin}(\mathrm{D}_{(01)}((4,2,4)_M))$
3	1	3	4	12	$6k$	$2k$	$3k$	k	$24k$	$\mathrm{Pin}(\mathrm{D}_{(02)}((6,2,3)_M))$
4	1	3	6	6	$3k$	k	k	k	$12k$	$\mathrm{Pin}((3,3,3)_M)$
5	1	4	4	8	$4k$	k	$2k$	k	$16k$	$\mathrm{Pin}((4,2,4)_M)$
6	1	6	4	6	$6k$	k	$3k$	$2k$	$24k$	$\mathrm{Pin}((6,2,3)_M)$
7	2	2	4	4	k	k	k	k	$8k$	$\mathrm{D}_{(01)}((4,2,4)_M)$
8	2	3	2	12	$3k$	$2k$	$6k$	k	$24k$	$\mathrm{Walsh}(\mathrm{D}_{(02)}((6,2,3)_M))$
9	2	4	2	8	$2k$	k	$4k$	k	$16k$	$\mathrm{Walsh}((4,2,4)_M)$
10	2	6	2	6	$3k$	k	$6k$	$2k$	$24k$	$\mathrm{Walsh}((6,2,3)_M)$
11	3	3	2	6	k	k	$3k$	k	$12k$	$\mathrm{Walsh}((3,3,3)_M)$
12	3	6	2	4	$2k$	k	$6k$	$3k$	$24k$	$\mathrm{Walsh}(\mathrm{D}_{(12)}((6,2,3)_M))$
13	4	4	2	4	k	k	$4k$	$2k$	$16k$	$\mathrm{Walsh}(\mathrm{D}_{(12)}((4,2,4)_M))$

Table 4.1: The bipartite-uniform hypermaps on the torus (up to duality). ($k = \det(M)$.)

Table 4.1 lists all possible values for the bipartite-type of a bipartite-uniform hypermap on the torus, up to duality. The hypermaps listed in lines 1-6 and 8-13 are obtained from uniform hypermaps by the Pin and Walsh constructions, and the hypermap in line 7 is dual of a uniform map of type $(4,2,4)$.

4.3 Chirality groups and chirality indices of the 2-restrictedly-regular hypermaps on the torus

In this Section we compute the chirality groups and the chirality indices of the 2-restrictedly-regular hypermaps on the torus. In Table 4.2 we display the chirality groups, chirality indices and closure covers of the restrictedly-regular hypermaps listed in Proposition 4.1.12. Table 4.3 lists the chirality groups, chirality indices and closure covers of the bipartite-regular hypermaps on the torus obtained by the Walsh and Pin constructions.

The following Lemma will be very useful in this section.

Lemma 4.3.1. *Let* $M := \begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

1. (a) *If* M *is* $\begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix}$ *or* $\begin{pmatrix} l & -l \\ m & m \end{pmatrix}$, *then* $(4,2,4)_M$ *has hypermap subgroup* $\langle (R_1R_2)^4, (R_2R_0)^2, [(R_2R_0)^2]^{R_1}, (R_0R_1)^4, X^aY^b, X^cY^d\rangle^{\Delta^{\hat{1}}}$;
   (b) *If* M *is* $\begin{pmatrix} l & -m \\ l & m \end{pmatrix}$ *or* $\begin{pmatrix} l & m \\ m & l \end{pmatrix}$, *then* $(4,2,4)_M$ *has hypermap subgroup* $\langle (R_1R_2)^4, (R_2R_0)^2, (R_0R_1)^4, X^aY^b, X^cY^d\rangle^{\Delta^1}$;
   (c) *If* M *is* $\begin{pmatrix} l & -m \\ m & l \end{pmatrix}$, *then* $(4,2,4)_M$ *has hypermap subgroup* $\langle (R_1R_2)^4, (R_2R_0)^2, (R_0R_1)^4, X^lY^m, X^{-m}Y^l\rangle^{\Delta^+}$;
   (d) *If* M *is* $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ *or* $\begin{pmatrix} k & -k \\ k & k \end{pmatrix}$, *then* $(4,2,4)_M$ *has hypermap subgroup* $\langle (R_1R_2)^4, (R_2R_0)^2, (R_0R_1)^4, X^aY^b, X^cY^d\rangle^{\Delta}$.

2. (a) *If* M *is* $\begin{pmatrix} l & -l-m \\ m & l \end{pmatrix}$, *then* $(6,2,3)_M$ *has hypermap subgroup* $\langle (R_1R_2)^6, (R_2R_0)^2, (R_0R_1)^3, X^lY^m, X^{-l-m}Y^l\rangle^{\Delta^+}$;
   (b) *If* M *is* $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ *or* $\begin{pmatrix} k & -2k \\ k & k \end{pmatrix}$, *then* $(6,2,3)_M$ *has hypermap subgroup* $\langle (R_1R_2)^6, (R_2R_0)^2, (R_0R_1)^3, X^aY^b, X^cY^d\rangle^{\Delta}$.

3. (a) *If* M *is* $\begin{pmatrix} l & -l-m \\ m & l \end{pmatrix}$, *then* $(3,3,3)_M$ *has hypermap subgroup* $\langle (R_1R_2)^3, (R_2R_0)^3, (R_0R_1)^3, X^lY^m, X^{-l-m}Y^l\rangle^{\Delta^+}$;
   (b) *If* M *is* $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ *is* $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ *or* $\begin{pmatrix} k & -2k \\ k & k \end{pmatrix}$, *then* $(3,3,3)_M$ *has hypermap subgroup* $\langle (R_1R_2)^3, (R_2R_0)^3, (R_0R_1)^3, X^aY^b, X^cY^d\rangle^{\Delta}$.

Proof. 1. Let Θ be Δ^1, $\Delta^{\hat{0}}$ or Δ^+, and $\mathcal{H} = (4,2,4)_M$ a Θ-regular hypermap with hypermap subgroup $H := \langle S\rangle^{\Delta}\langle X^aY^b, X^cY^d\rangle$, where $S := \{(R_1R_2)^4, (R_2R_0)^2, (R_0R_1)^4\}$.

Since $R_1 \notin \Delta^{\hat{1}}$, by 1. of Lemma 1.9.2,

$$\langle S\rangle^{\Delta} = \langle S, S^{R_1}\rangle^{\Delta^{\hat{1}}} = \langle S, [(R_1R_2)^4]^{-1}, [(R_2R_0)^2]^{R_1}, [(R_0R_1)^4]^{-1}\rangle^{\Delta^{\hat{1}}} = \langle S, [(R_2R_0)^2]^{R_1}\rangle^{\Delta^{\hat{1}}}.$$

When Θ is Δ^+ or Δ^1, $R_0 \notin \Theta$, but $R_2R_0 \in \Theta$, so

$$\langle S\rangle^{\Delta} = \langle S, S^{R_0}\rangle^{\Theta} = \langle S, ([(R_1R_2)^4]^{-1})^{R_2R_0}, [(R_2R_0)^2]^{-1}, [(R_0R_1)^4]^{-1}\rangle^{\Theta} = \langle S\rangle^{\Theta}.$$

Let $P = S \cup \{X^aY^b, X^cY^d\}$ and $Q = P \cup \{[(R_1R_2)^4]^{-1}, [(R_2R_0)^2]^{R_1}\}$. If Θ is Δ, Δ^+ or Δ^1, and $\mathcal{H}$ is Θ-regular, then $H \subseteq \langle P\rangle^{\Theta}$. On the other hand, since H is a normal subgroup of Θ containing P, H also contains $\langle P\rangle^{\Theta}$. Similarly, if $\mathcal{H}$ is $\Delta^{\hat{0}}$-regular, then $H = \langle Q\rangle^{\Delta^{\hat{0}}}$.
2. and 3. are similar to 1. □

Having in mind that $N_iX_i \leftrightharpoons N_iY_i$ and

1. (a) $N_4Y_4 = N_4{X_4}^{R_1}$ and $N_4X_4 = N_4(R_0R_1R_2R_1)$,
   (b) $N_4{X_4}^{-1}Y_4 = N_4(X_4Y_4)^{R_0}$ and $N_4X_4Y_4 = N_4(R_0R_1R_2)^2$,
2. (a) $N_6Y_6 = N_6{X_6}^{R_1}$ and $N_6Y_6 = N_6R_0R_1R_0R_2R_1R_2 = N_6(R_0R_1R_2)^2$,
   (b) $N_6{X_6}^{-2}Y_6 = N_6(X_6Y_6)^{R_0}$ and $N_6X_6Y_6 = N_6(R_0R_1R_2R_1R_2)^2$,
3. (a) $N_3Y_3 = N_3{X_3}^{R_1}$ and $N_3X_3 = N_3(R_2R_1R_0R_1)$,
   (b) $N_3{X_3}^{-2}Y_3 = N_3(X_3Y_3)^{R_2}$ and $N_3X_3Y_3 = N_3(R_2R_1R_0)^2$,

we can find more convenient presentations for the hypermap subgroups of the regular hypermaps on the torus.

Corollary 4.3.2.

1. *(a)* $(4,2,4)_{\left(\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}\right)}$ *has hypermap subgroup* $\langle (R_1R_2)^4, (R_2R_0)^2, (R_0R_1)^4, (R_0R_1R_2R_1)^k\rangle^\Delta$;

   (b) $(4,2,4)_{\left(\begin{smallmatrix} k & \text{-}k \\ k & k \end{smallmatrix}\right)}$ *has hypermap subgroup* $\langle (R_1R_2)^4, (R_2R_0)^2, (R_0R_1)^4, (R_0R_1R_2)^{2k}\rangle^\Delta$.

2. *(a)* $(6,2,3)_{\left(\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}\right)}$ *has hypermap subgroup* $\langle (R_1R_2)^6, (R_2R_0)^2, (R_0R_1)^3, (R_0R_1R_2)^{2k}\rangle^\Delta$;

   (b) $(6,2,3)_{\left(\begin{smallmatrix} k & \text{-}2k \\ k & k \end{smallmatrix}\right)}$ *has hypermap subgroup* $\langle (R_1R_2)^6, (R_2R_0)^2, (R_0R_1)^3, (R_0R_1R_2R_1R_2)^{2k}\rangle^\Delta$.

3. *(a)* $(3,3,3)_{\left(\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}\right)}$ *has hypermap subgroup* $\langle (R_1R_2)^3, (R_2R_0)^3, (R_0R_1)^3, (R_2R_1R_0R_1)^k\rangle^\Delta$;

   (b) $(3,3,3)_{\left(\begin{smallmatrix} k & \text{-}2k \\ k & k \end{smallmatrix}\right)}$ *has hypermap subgroup* $\langle (R_1R_2)^3, (R_2R_0)^3, (R_0R_1)^3, (R_2R_1R_0)^{2k}\rangle^\Delta$.

4.3.1 Chirality groups and chirality indices of the orientably-regular hypermaps on the torus

Up to duality, there are 3 families of orientably-regular hypermaps on the torus: $(4,2,4)_{\left(\begin{smallmatrix} l & -m \\ m & l \end{smallmatrix}\right)}$, $(6,2,3)_{\left(\begin{smallmatrix} l & -l-m \\ m & l \end{smallmatrix}\right)}$, $(3,3,3)_{\left(\begin{smallmatrix} l & -l-m \\ m & l \end{smallmatrix}\right)}$, with $l, m \in \mathbb{N}$. The chirality groups and chirality indices of the first two families (that is, the families of maps) have been computed in [4].

- $\mathcal{Q} = (4,2,4)_{\left(\begin{smallmatrix} l & -m \\ m & l \end{smallmatrix}\right)}$ has hypermap subgroup
  $Q = \langle (R_1R_2)^4, (R_2R_0)^2, (R_0R_1)^4, X^lY^m, X^{-m}Y^l\rangle^{\Delta^+}$.
  Let $k := \gcd(l,m)$. Then

$$\begin{aligned}\Upsilon = Q^\Delta/Q = QQ^{R_1}/Q &= \langle Q(X^lY^m)^{R_1}, Q(X^{-m}Y^l)^{R_1}\rangle^{\Delta^+/Q}\\ &= \langle QX^mY^l, QX^lY^{-m}\rangle^{\Delta^+/Q}\\ &= \langle QX^mY^l, QX^lY^{-m}\rangle\\ &= \langle QX^{2m}, QX^{2l}\rangle\\ &= \langle QX^{2k}\rangle.\end{aligned}$$

  Because QX has order $(l^2+m^2)/k$ in Δ^+/Q, QX^{2k} has order $\frac{(l^2+m^2)/k}{\gcd(2k,(l^2+m^2)/k)}$.

If l/k and m/k are both odd, that is, if $2k \mid l-m$, then $\gcd(2k,(l^2+m^2)/k)=2k$, $\Upsilon \cong C_{(l^2+m^2)/2k^2}$ and $\iota=(l^2+m^2)/2k^2$. In addition

$$\begin{pmatrix} l & -m \\ m & l \end{pmatrix} = \begin{pmatrix} k & -k \\ k & k \end{pmatrix} \begin{pmatrix} \frac{l+m}{2k} & \frac{l-m}{2k} \\ \frac{-l+m}{2k} & \frac{l+m}{2k} \end{pmatrix} \text{ and } \mathcal{Q}^\Delta = (4,2,4)_{\left(\begin{smallmatrix} k & -k \\ k & k \end{smallmatrix}\right)}.$$

If l/k and m/k are not both odd, that is, if $2k \nmid l-m$, then $\gcd(2k,(l^2+m^2)/k)=k$, $\Upsilon \cong C_{(l^2+m^2)/k^2}$ and $\iota=(l^2+m^2)/k^2$. In addition

$$\begin{pmatrix} l & -m \\ m & l \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} l/k & -m/k \\ m/k & l/k \end{pmatrix} \text{ and } \mathcal{Q}^\Delta = (4,2,4)_{\left(\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}\right)}.$$

- $\mathcal{Q}=(6,2,3)_{\left(\begin{smallmatrix} l & -l-m \\ m & l \end{smallmatrix}\right)}$ has hypermap subgroup
  $Q=\langle (R_1R_2)^6,(R_2R_0)^2,(R_0R_1)^3,X^lY^m,X^{-l-m}Y^l\rangle^{\Delta^+}$.

Let $k:=\gcd(l,m)$. Then

$$\begin{aligned}
\Upsilon = Q^\Delta/Q = QQ^{R_1}/Q &= \langle Q(X^lY^m)^{R_1}, Q(X^{-l-m}Y^l)^{R_1}\rangle^{\Delta^+/Q} \\
&= \langle QX^mY^l, QX^lY^{-l-m}\rangle^{\Delta^+/Q} \\
&= \langle QX^mY^l, QX^lY^{-l-m}\rangle \\
&= \langle QX^{l+2m}, QX^{l-m}\rangle \\
&= \langle QX^{3l}, QX^{l-m}\rangle \\
&= \langle QX^{\gcd(3l,l-m)}\rangle.
\end{aligned}$$

We note that QX has order $(l^2+lm+m^2)/k$ in Δ^+/Q.
If $3k \mid l-m$, then $\gcd(3l,l-m)=3k$, $3k^2 \mid l^2+lm+m^2=(l-m)(l+2m)+3m^2$ and QX^{3k} has order

$$\frac{(l^2+lm+m^2)/k}{\gcd(3k,(l^2+lm+m^2)/k)} = \frac{(l^2+lm+m^2)/k}{3k} = (l^2+lm+m^2)/3k^2.$$

Furthermore $\Upsilon=\langle QX^{3k}\rangle \cong C_{(l^2+lm+m^2)/3k^2}$, $\iota=(l^2+lm+m^2)/3k^2$,

$$\begin{pmatrix} l & -l-m \\ m & l \end{pmatrix} = \begin{pmatrix} k & -2k \\ k & k \end{pmatrix} \begin{pmatrix} \frac{l+2m}{3k} & \frac{l-m}{3k} \\ \frac{-l+m}{3k} & \frac{2l+m}{3k} \end{pmatrix} \text{ and } \mathcal{Q}^\Delta = (6,2,3)_{\left(\begin{smallmatrix} k & -2k \\ k & k \end{smallmatrix}\right)}.$$

If $3k \nmid l-m$, then $\gcd(3l,l-m)=k$ and QX^k has order

$$\frac{(l^2+lm+m^2)/k}{\gcd(k,(l^2+lm+m^2)/k)} = \frac{(l^2+lm+m^2)/k}{k} = (l^2+lm+m^2)/k^2.$$

Furthermore $\Upsilon=\langle QX^k\rangle \cong C_{(l^2+lm+m^2)/k^2}$, $\iota=(l^2+lm+m^2)/k^2$,

$$\begin{pmatrix} l & -l-m \\ m & l \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} l/k & -(l+m)/k \\ m/k & l/k \end{pmatrix} \text{ and } \mathcal{Q}^\Delta = (6,2,3)_{\left(\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}\right)}.$$

- $\mathcal{Q} = (3,3,3)_{\left(\begin{smallmatrix} l & -l-m \\ m & l \end{smallmatrix}\right)}$ has hypermap subgroup
$Q = \langle (R_1R_2)^3, (R_2R_0)^3, (R_0R_1)^3, X^lY^m, X^{-l-m}Y^l \rangle^{\Delta^+}$.
Let $M = \begin{pmatrix} l & -l-m \\ m & l \end{pmatrix}$, $\mathcal{H} = (6,2,3)_M$ and $\varphi_W^* := \overline{(02)}|_{\Delta^{\hat{2}}} \circ \varphi_W : \Delta^{\hat{2}} \to \Delta$. Then $\mathrm{D}_{(02)}(\mathcal{H}) \cong \mathrm{Walsh}(\mathcal{Q})$ and $H = Q{\varphi_W^*}^{-1}$. We claim that $H^{\Delta^{\hat{2}}} = H^\Delta$ and $\Upsilon(\mathcal{Q}) \cong \Upsilon(\mathcal{H})$. By Corollary 4.1.7, $R_2R_0 \in \mathrm{N}_\Delta(H)$. According to 4. of Proposition A.1.6, $R_2R_0 \in \mathrm{N}_\Delta(H^{\Delta^{\hat{2}}})$, so $\Delta = \langle \Delta^{\hat{2}}, R_2R_0 \rangle \subseteq \mathrm{N}_\Delta(H^{\Delta^{\hat{2}}})$, that is, $\mathrm{N}_\Delta(H^{\Delta^{\hat{2}}}) = \Delta$, or equivalently, $\mathrm{N}_\Delta(H^{\Delta^{\hat{2}}}) \lhd \Delta$. Finally, using 2. of Proposition A.1.6, we have $H^{\Delta^{\hat{2}}} = (H^{\Delta^{\hat{2}}})^\Delta = H^\Delta$. Since φ_W^* is an epimorphism, $H^{\Delta^{\hat{2}}} = Q^\Delta {\varphi_W^*}^{-1}$ (by Corollary A.1.9) and
$$\Upsilon(\mathcal{H}) = H^\Delta/H = H^{\Delta^{\hat{2}}}/H = Q^\Delta{\varphi_W^*}^{-1}/Q{\varphi_W^*}^{-1} \cong Q^\Delta/Q = \Upsilon(\mathcal{Q}).$$
Let $k := \gcd(l,m)$. Like in the previous case $\Upsilon = \langle QX^{\gcd(3l,l-m)} \rangle$.
If $3k \mid l-m$, then $\Upsilon \cong C_{(l^2+lm+m^2)/3k^2}$, $\iota = (l^2+lm+m^2)/3k^2$ and $\mathcal{Q}^\Delta = (3,3,3)_{\left(\begin{smallmatrix} k & -2k \\ k & k \end{smallmatrix}\right)}$.
If $3k \nmid l-m$, then $\Upsilon \cong C_{(l^2+lm+m^2)/k^2}$, $\iota = (l^2+lm+m^2)/k^2$ and $\mathcal{Q}^\Delta = (3,3,3)_{\left(\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}\right)}$.

4.3.2 Chirality groups and chirality indices of the pseudo-orientably-regular hypermaps on the torus

There are 2 families of pseudo-orientably-regular hypermaps on the torus:
the duals of $(4,2,4)_{\left(\begin{smallmatrix} l & -m \\ l & m \end{smallmatrix}\right)}$ and $(4,2,4)_{\left(\begin{smallmatrix} l & m \\ m & l \end{smallmatrix}\right)}$, with $l, m \in \mathbb{N}$.

- $\mathcal{P} = (4,2,4)_{\left(\begin{smallmatrix} l & -m \\ l & m \end{smallmatrix}\right)}$ has hypermap subgroup
$P = \langle (R_1R_2)^4, (R_2R_0)^2, (R_0R_1)^4, X^lY^l, X^{-m}Y^m \rangle^{\Delta^1}$.
Let $k := \gcd(l,m)$. Then
$$\begin{aligned} \Upsilon = P^\Delta/P = PP^{R_0}/P &= \langle P(X^lY^l)^{R_0}, P(X^{-m}Y^m)^{R_0} \rangle^{\Delta^1/P} \\ &= \langle PX^{-l}Y^l, PX^mY^m \rangle^{\Delta^1/P} \\ &= \langle P(X^{-1}Y)^l, P(XY)^m \rangle^{\Delta^1/P} \\ &= \langle P(X^{-1}Y)^k, P(XY)^k \rangle^{\Delta^1/P} \\ &= \langle PX^{-k}Y^k, PX^kY^k \rangle^{\Delta^1/P} \\ &= \langle PX^{-k}Y^k, PX^kY^k \rangle \\ &\cong C_{l/k} \times C_{m/k}. \end{aligned}$$
Since $\gcd(l/k, m/k)$ is 1, $\Upsilon \cong C_{l/k} \times C_{m/k} \cong C_{lm/k^2}$ and $\iota = lm/k^2$. In addition
$$\begin{pmatrix} l & -m \\ l & m \end{pmatrix} = \begin{pmatrix} k & -k \\ k & k \end{pmatrix} \begin{pmatrix} l/k & 0 \\ 0 & m/k \end{pmatrix} \text{ and } \mathcal{P}^\Delta = (4,2,4)_{\left(\begin{smallmatrix} k & -k \\ k & k \end{smallmatrix}\right)}.$$

- $\mathcal{P} = (4,2,4)_{\left(\begin{smallmatrix} l & m \\ m & l \end{smallmatrix}\right)}$ has hypermap subgroup
$P = \langle (R_1R_2)^4, (R_2R_0)^2, (R_0R_1)^4, X^lY^m, X^mY^l \rangle^{\Delta^1}$.

Let $k := \gcd(l, m)$. Then

$$\begin{aligned}\Upsilon = P^{\Delta}/P = PP^{R_0}/P &= \langle P(X^lY^m)^{R_0}, P(X^mY^l)^{R_0}\rangle^{\Delta^1/P}\\ &= \langle PX^{-l}Y^m, PX^{-m}Y^l\rangle^{\Delta^1/P}\\ &= \langle PX^{-l}Y^m, PX^{-m}Y^l\rangle\\ &= \langle PX^{-2l}, PX^{-2m}\rangle\\ &= \langle PX^{2k}\rangle.\end{aligned}$$

Because PX has order $|l^2-m^2|/k$ in Δ^1/P, PX^{2k} has order

$$\frac{|l^2-m^2|/k}{\gcd(2k, |l^2-m^2|/k)} = \frac{|l^2-m^2|/k}{k\gcd(2, |l^2-m^2|/k^2)} = \frac{|l^2-m^2|/k^2}{\gcd(2, |l^2-m^2|/k^2)}.$$

If l/k and m/k are both odd, then $\Upsilon = C_{|l^2-m^2|/2k^2}$, $\iota = |l^2-m^2|/2k^2$,

$$\begin{pmatrix} l & m\\ m & l\end{pmatrix} = \begin{pmatrix} k & -k\\ k & k\end{pmatrix}\begin{pmatrix} \frac{l+m}{2k} & \frac{l-m}{2k}\\ \frac{-l+m}{2k} & \frac{l+m}{2k}\end{pmatrix} \text{ and } \mathcal{P}^{\Delta} = (4,2,4)_{\left(\begin{smallmatrix} k & -k\\ k & k\end{smallmatrix}\right)};$$

else if l/k and m/k are not simultaneously odd, then $\Upsilon = C_{|l^2-m^2|/k^2}$, $\iota = |l^2-m^2|/k^2$,

$$\begin{pmatrix} l & m\\ m & l\end{pmatrix} = \begin{pmatrix} k & 0\\ 0 & k\end{pmatrix}\begin{pmatrix} l/k & m/k\\ m/k & l/k\end{pmatrix} \text{ and } \mathcal{P}^{\Delta} = (4,2,4)_{\left(\begin{smallmatrix} k & 0\\ 0 & k\end{smallmatrix}\right)}.$$

4.3.3 Chirality groups and chirality indices of the bipartite-regular hypermaps on the torus

We have seen in Section 4.2 that there are 3 kinds of bipartite-regular hypermaps on the torus: the duals of $\Delta^{\hat{1}}$-regular maps of type $(4,2,4)$, and the hypermaps obtained from regular maps by the Walsh and Pin constructions.

We recall that $(4,2,4)_M$ and $\mathrm{D}_{(01)}((4,2,4)_M)$ have the same chirality group.

- $\mathcal{B} = (4,2,4)_{\left(\begin{smallmatrix} l & 0\\ 0 & m\end{smallmatrix}\right)}$ has hypermap subgroup $\langle (R_1R_2)^4, (R_2R_0)^2, [(R_2R_0)^2]^{R_1}, (R_0R_1)^4, X^l, Y^m\rangle^{\Delta^{\hat{1}}}$.
Let $k := \gcd(l, m)$. Then

$$\begin{aligned}\Upsilon = B^{\Delta}/B = BB^{R_1}/B &= \langle B(X^l)^{R_1}, B(Y^m)^{R_1}\rangle^{\Delta^{\hat{1}}/B}\\ &= \langle BY^l, BX^m\rangle^{\Delta^{\hat{1}}/B}\\ &= \langle BY^k, BX^k\rangle^{\Delta^{\hat{1}}/B}\\ &= \langle BY^k, BX^k\rangle\\ &\cong C_{l/k}\times C_{m/k}.\end{aligned}$$

Since $\gcd(l/k, m/k)$ is 1, $\Upsilon \cong C_{l/k}\times C_{m/k} \cong C_{lm/k^2}$ and $\iota = lm/k^2$. In addition

$$\begin{pmatrix} l & 0\\ 0 & m\end{pmatrix} = \begin{pmatrix} k & 0\\ 0 & k\end{pmatrix}\begin{pmatrix} l/k & 0\\ 0 & m/k\end{pmatrix} \text{ and } \mathcal{B}^{\Delta} = (4,2,4)_{\left(\begin{smallmatrix} k & 0\\ 0 & k\end{smallmatrix}\right)}.$$

- $\mathcal{B} = (4,2,4)_{\left(\begin{smallmatrix} l & -l \\ m & m \end{smallmatrix}\right)}$ has hypermap subgroup $\langle (R_1R_2)^4, (R_2R_0)^2, [(R_2R_0)^2]^{R_1}, (R_0R_1)^4, X^lY^m, X^{-l}Y^m \rangle^{\Delta^{\hat{1}}}$.
Let $k := \gcd(l,m)$. Then

$$\begin{aligned}
\Upsilon = B^{\Delta}/B = BB^{R_1}/B &= \langle B(X^lY^m)^{R_1}, B(X^{-l}Y^m)^{R_1} \rangle^{\Delta^{\hat{1}}/B} \\
&= \langle BX^mY^l, BX^mY^{-l} \rangle^{\Delta^{\hat{1}}/B} \\
&= \langle BX^mY^l, BX^mY^{-l} \rangle \\
&= \langle BX^{l+m}Y^{l+m}, BX^{-l+m}Y^{-l+m} \rangle \\
&= \langle B(XY)^{l+m}, B(XY)^{-l+m} \rangle \\
&= \langle B(XY)^{\gcd(l+m,-l+m)} \rangle.
\end{aligned}$$

If l/k and m/k are both odd, then $\gcd(l+m,-l+m) = \gcd(l+m,2m) = 2k$, BXY has order lm/k in $\Delta^{\hat{1}}/B$, $BX^{2k}Y^{2k} = B(XY)^{2k}$ has order

$$\frac{lm/k}{\gcd(2k, lm/k)} = \frac{lm/k}{k} = lm/k^2,$$

$\Upsilon = \langle BX^kY^k \rangle = C_{lm/k^2}$ and $\iota = lm/k^2$. In addition

$$\begin{pmatrix} l & -l \\ m & m \end{pmatrix} = \begin{pmatrix} k & -k \\ k & k \end{pmatrix} \begin{pmatrix} \frac{l+m}{2k} & \frac{l+m}{2k} \\ \frac{-l+m}{2k} & \frac{-l+m}{2k} \end{pmatrix} \text{ and } \mathcal{B}^{\Delta} = (4,2,4)_{\left(\begin{smallmatrix} k & -k \\ k & k \end{smallmatrix}\right)}.$$

If l/k and m/k are not both odd, then $\gcd(l+m,-l+m) = \gcd(l+m,2m) = k$, BXY has order $2lm/k$ in $\Delta^{\hat{1}}/B$, $BX^kY^k = B(XY)^k$ has order

$$\frac{2lm/k}{\gcd(k, 2lm/k)} = \frac{2lm/k}{k} = 2lm/k^2,$$

$\Upsilon = \langle BX^{2k}Y^{2k} \rangle = C_{2lm/k^2}$ and $\iota = 2lm/k^2$. In addition

$$\begin{pmatrix} l & -l \\ m & m \end{pmatrix} = \begin{pmatrix} k & -k \\ k & k \end{pmatrix} \begin{pmatrix} l/k & m/k \\ -l/k & m/k \end{pmatrix} \text{ and } \mathcal{B}^{\Delta} = (4,2,4)_{\left(\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}\right)}.$$

As a by-product of these calculations, we get the following result:

Theorem 4.3.3. *The chirality groups of 2-restrictedly-regular uniform hypermaps on the torus are cyclic groups.*

Finally, we compute the chirality groups and chirality indices of the bipartite-regular hypermaps on the torus obtained via the Walsh and Pin constructions.

Recall that the hypermap subgroup R of a regular hypermap $\mathcal{R}$ on the torus of type (l,m,n) is the normal closure in Δ of a set with 4 elements containing $(R_1R_2)^l$, $(R_2R_0)^m$ and $(R_0R_1)^n$. Indeed, $R = \langle T \rangle^{\Delta}$, where $T = \{(R_1R_2)^l, (R_2R_0)^m, (R_0R_1)^n, w\}$, and w is, up to duality, $(R_0R_1R_2R_1)^k$, $(R_0R_1R_2)^{2k}$ or $(R_0R_1R_2R_1R_2)^{2k}$.

Chirality groups and chirality indices of $\mathcal{B} = \mathrm{Walsh}(\mathcal{R})$

The bipartite-regular hypermaps on the torus obtained by the Walsh construction are listed, up to duality, in cases 8-13 of Table 4.1. We use the notations of Proposition 1.9.6.

Remark 4.3.4. Let $\mathcal{B} = \mathrm{Walsh}(\mathcal{R})$ and ι the chirality index of $\mathcal{B}$. If $\mathcal{K}$ is a hypermap covered by $\mathcal{H}$ such that Walsh($\mathcal{K}$) is regular and has $|\Omega_{\mathcal{B}}|/\iota$ flags, then $\mathcal{B}^\Delta = \mathrm{Walsh}(\mathcal{K})$.

Note that if $\{i,j,k\} = \{0,1,2\}$ and R is a normal subgroup of Δ containing $(R_iR_jR_k)^2$, then R also contains $(R_iR_jR_k)^2\alpha_W$: if $j=2$, then $(R_iR_jR_k)^2\alpha_W = (R_kR_jR_i)^2 = [(R_iR_jR_k)^2]^{-1}$, else, if $j \neq 2$, then $(R_iR_jR_k)^2\alpha_W = [(R_kR_jR_i)^2]^{R_2} = ([(R_iR_jR_k)^2]^{-1})^{R_2}$.

- Case **8**: $\mathcal{B} = \mathrm{Walsh}(\mathrm{D}_{(02)}((6,2,3)_M))$, $d_1 = 1$.
  Corollary 1.9.7 implies that $\Upsilon(\mathcal{B}) \cong \Delta^+/R \cong \mathrm{Aut}^+(\mathcal{R})$.
  If $M = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, then $\iota = |\Omega_{\mathcal{R}}|/2 = 12k^2/2 = 6k^2$.
  If $M = \begin{pmatrix} k & -2k \\ k & k \end{pmatrix}$, then $\iota = |\Omega_{\mathcal{R}}|/2 = 36k^2/2 = 18k^2$.
  In both cases $\mathcal{B}^\Delta$ is $\mathcal{S}_2$.

- Case **9**: $\mathcal{B} = \mathrm{Walsh}((4,2,4)_M)$, $d_1 = 2$.
  If $M = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, then $S = \{(R_0R_1R_2R_1)^k\}$,
  $R(R_0R_1R_2R_1)^k\alpha_W = R(R_1R_0R_2R_0)^k = R(R_1R_2)^k$ and

  $$\Upsilon(\mathcal{B}) \;\cong\; \langle R(R_1R_2)^2, R(R_2R_0)^2, R(R_1R_2)^k\rangle^{\Delta/R} = \langle R(R_1R_2)^{\gcd(2,k)}\rangle^{\Delta/R}.$$

  Note that $RX^2 = R[(R_1R_2)^2]^{R_0}(R_1R_2)^{-2}$ and $RY^2 = R(X^2)^{R_1} = R[(R_1R_2)^2]^{R_0R_1}(R_1R_2)^2$ are in $\langle R(R_1R_2)^{\gcd(2,k)}\rangle^{\Delta/R}$.
  When $2 \nmid k$,

  $$\Upsilon(\mathcal{B}) \cong \langle RR_1R_2\rangle^{\Delta/R} = \langle RR_1R_2, R(R_1R_2)^{R_0}, R(R_1R_2)^{R_0R_1}\rangle.$$

  Then $\langle RR_1R_2\rangle^{\Delta/R}$ contains $RX = (RX^2)^{\frac{k+1}{2}}$ and $RR_0R_1 = RXR_1R_2$. Therefore $\Upsilon(\mathcal{B}) \cong \Delta^+/R \cong \mathrm{Aut}^+(\mathcal{R})$, $\iota = |\Omega_{\mathcal{R}}|/2 = 8k^2/2 = 4k^2$ and $\mathcal{B}^\Delta$ is $\mathcal{S}_2$.
  When $2 \mid k$,

  $$\begin{aligned} \Upsilon(\mathcal{B}) &\cong \langle R(R_1R_2)^2\rangle^{\Delta/R} \\ &= \langle R(R_1R_2)^2, R[(R_1R_2)^2]^{R_0}, R[(R_1R_2)^2]^{R_0R_1}\rangle \\ &= \langle R(R_1R_2)^2, RX^2, RY^2\rangle \\ &\cong (C_{k/2} \times C_{k/2}) \rtimes C_2, \end{aligned}$$

  $\iota = k^2/2$ and $\mathcal{B}^\Delta$ is $\mathcal{P}_8 = \mathrm{Walsh}(\mathcal{P}4)$.
  If $M = \begin{pmatrix} k & -k \\ k & k \end{pmatrix}$, then $S = \{(R_0R_1R_2)^{2k}\}$, $R(R_0R_1R_2)^{2k}\alpha_W = R$ and

  $$\begin{aligned} \Upsilon(\mathcal{B}) &\cong \langle R(R_1R_2)^2, R(R_2R_0)^2\rangle^{\Delta/R} \\ &= \langle R(R_1R_2)^2\rangle^{\Delta/R} \\ &= \langle R(R_1R_2)^2, R[(R_1R_2)^2]^{R_0}, R[(R_1R_2)^2]^{R_0R_1}\rangle \\ &= \langle R(R_1R_2)^2, RX^2, RY^2\rangle \\ &= \langle R(R_1R_2)^2, RX^2, RX^2Y^2\rangle. \end{aligned}$$

When $2 \nmid k$, $RXY = (RX^2Y^2)^{\frac{k+1}{2}}$, so

$$\begin{aligned} \Upsilon(\mathcal{B}) &\cong \langle R(R_1R_2)^2, RX^2, RX^2Y^2\rangle \\ &= \langle R(R_1R_2)^2, RX^2, RXY\rangle \\ &\cong (C_k \times C_k) \rtimes C_2, \end{aligned}$$

$\iota = 2k^2$ and $\mathcal{B}^\Delta$ is $\mathcal{P}_4 = \text{Walsh}(\mathcal{P}_2)$.
When $2 \mid k$,

$$\begin{aligned} \Upsilon(\mathcal{B}) &\cong \langle R(R_1R_2)^2, RX^2, RX^2Y^2\rangle \\ &\cong (C_k \times C_{k/2}) \rtimes C_2, \end{aligned}$$

$\iota = k^2$ and $\mathcal{B}^\Delta$ is $\mathcal{P}_8 = \text{Walsh}(\mathcal{P}_4)$.

- Case **10**: $\mathcal{B} = \text{Walsh}((6,2,3)_M)$, $d_1 = \gcd(l,m) = 2$.
If $M = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, then $S = \{(R_0R_1R_2)^{2k}\}$, $R(R_0R_1R_2)^{2k}\alpha_W = R$, and

$$\begin{aligned} \Upsilon(\mathcal{B}) &\cong \langle R(R_1R_2)^2, R(R_2R_0)^2\rangle^{\Delta/R} \\ &= \langle R(R_1R_2)^2\rangle^{\Delta/R} \\ &= \langle R(R_1R_2)^2, R[(R_1R_2)^2]^{R_0}\rangle. \end{aligned}$$

Since $RXY = R[(R_1R_2)^2]^{R_0}(R_1R_2)^2$ and $RX^{-2}Y = R(R_1R_2)^2[(R_1R_2)^2]^{R_0}$, RX^3 and RY^3 are in $\langle R(R_1R_2)^2\rangle^{\Delta/R}$, and $\Upsilon(\mathcal{B}) \cong \langle R(R_1R_2)^2, RXY, RX^{-2}Y\rangle$.
When $3 \nmid k$, RX and RY are in $\langle R(R_1R_2)^2\rangle^{\Delta/R}$, as well as $RR_0R_1 = RX[(R_1R_2)^2]^{-1}$. Then

$$\begin{aligned} \Upsilon(\mathcal{B}) &\cong \langle R(R_1R_2)^2, RXY, RX^{-2}Y\rangle \\ &= \langle R(R_1R_2)^2, RX^3, RXY\rangle \\ &= \langle R(R_1R_2)^2, RX, RXY\rangle \\ &= \langle R(R_1R_2)^2, RX, RY\rangle \\ &\cong (C_k \times C_k) \rtimes C_3, \end{aligned}$$

$\iota = 3k^2$ and $\mathcal{B}^\Delta$ is $\mathcal{P}_2 = \text{Walsh}(\mathcal{P}_1)$.
When $3 \mid k$,

$$\begin{aligned} \Upsilon(\mathcal{B}) &\cong \langle R(R_1R_2)^2, RXY, RX^{-2}Y\rangle \\ &= \langle R(R_1R_2)^2, RXY, RX^3\rangle \\ &\cong (C_k \times C_{k/3}) \rtimes C_3, \end{aligned}$$

$\iota = k^2$ and $\mathcal{B}^\Delta$ is $\mathcal{P}_6 = \text{Walsh}(\mathcal{P}_3)$.
If $M = \begin{pmatrix} k & -2k \\ k & k \end{pmatrix}$, then $S = \{(R_0R_1R_2R_1R_2)^{2k}\}$,
$R(R_0R_1R_2R_1R_2)^{2k}\alpha_W = R(R_1R_0R_2R_0R_2)^{2k} = RR_1{}^{2k} = R$,

$$\begin{aligned} \Upsilon(\mathcal{B}) &\cong \langle R(R_1R_2)^2, R(R_2R_0)^2\rangle^{\Delta/R} = \langle R(R_1R_2)^2\rangle^{\Delta/R} \\ &= \langle R(R_1R_2)^2, R[(R_1R_2)^2]^{R_0}\rangle \\ &= \langle R(R_1R_2)^2, RXY, RX^{-2}Y\rangle \\ &\cong (C_k \times C_k) \rtimes C_3, \end{aligned}$$

$\iota = 3k^2$ and $\mathcal{B}^\Delta$ is $\mathcal{P}_6 = \text{Walsh}(\mathcal{P}_3)$.

- Case **11**: $\mathcal{B} = \text{Walsh}((3,3,3)_M) = \text{D}_{(02)}((6,2,3)_M)$, $\Upsilon(\mathcal{B}) \cong 1$ and $\iota = 1$. Then $\mathcal{B}$ is regular and $\mathcal{B}^\Delta = \mathcal{B}$.

- Case **12**: $\mathcal{B} = \text{Walsh}(\text{D}_{(12)}((6,2,3)_M))$, $d_1 = \gcd(l,m) = 3$.
Let $\overline{X} = X\overline{(12)} = R_0R_2R_1R_2R_1R_2$ and $\overline{Y} = Y\overline{(12)} = R_2R_0R_2R_1R_2R_1$. Because $R\overline{(12)}X \leftrightharpoons R\overline{(12)}Y$, $R\overline{X} \leftrightharpoons R\overline{Y}$.
If $M = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, then $S = \{(R_0R_1R_2)^{2k}\overline{(12)}\} = \{(R_0R_2R_1)^{2k}\}$, $R(R_0R_2R_1)^{2k}\alpha_W = R$.
Because $R\overline{X}^2 = R[(R_1R_2)^3]^{R_0}(R_1R_2)^3$, $R\overline{Y}^2 = R[(R_1R_2)^3]^{R_0R_2}(R_1R_2)^3$,
$$\begin{aligned}\Upsilon(\mathcal{B}) &\cong \langle R(R_1R_2)^3, R(R_2R_0)^3\rangle^{\Delta/R} \\ &= \langle R(R_1R_2)^3\rangle^{\Delta/R} \\ &= \langle R(R_1R_2)^3, R[(R_1R_2)^3]^{R_0}, R[(R_1R_2)^3]^{R_0R_2}\rangle \\ &= \langle R(R_1R_2)^3, R\overline{X}^2, R\overline{Y}^2\rangle.\end{aligned}$$
When $2 \nmid k$, $R\overline{X} = (R\overline{X}^2)^{\frac{k+1}{2}}$ and $R\overline{Y} = (R\overline{Y}^2)^{\frac{k+1}{2}}$ are in $\langle R(R_1R_2)^3\rangle^{\Delta/R}$,
$$\Upsilon(\mathcal{B}) \cong \langle R(R_1R_2)^3, R\overline{X}, R\overline{Y}\rangle \cong (C_k \times C_k) \rtimes C_2,$$
$\iota = 2k^2$ and $\mathcal{B}^\Delta$ is $\text{D}_{(02)}(\mathcal{P}_3) = \text{Walsh}(\mathcal{D}_3)$.
When $2 \mid k$,
$$\Upsilon(\mathcal{B}) \cong \langle R(R_1R_2)^3, R\overline{X}^2, R\overline{Y}^2\rangle \cong (C_{k/2} \times C_{k/2}) \rtimes C_2,$$
$\iota = k^2/2$ and $\mathcal{B}^\Delta$ is $\mathcal{C} = \text{Walsh}(\text{D}_{(12)}(\mathcal{T}))$.
If $M = \begin{pmatrix} k & -2k \\ k & k \end{pmatrix}$, then $S = \{(R_0R_1R_2R_1R_2)^{2k}\overline{(12)}\} = \{(R_0R_2R_1R_2R_1)^{2k}\}$ and
$R(R_0R_2R_1R_2R_1)^{2k}\alpha_W = R(R_1R_2R_0R_2R_0)^{2k} = R(R_1R_0R_2)^{2k} = R(R_0R_1R_2)^{2k} = R(\overline{X}\,\overline{Y}^{-1})^k$.
Since $R\overline{X} = R(\overline{X}\,\overline{Y}^{-1})^{R_1R_2}$ and $R\overline{Y} = R(\overline{X}\,\overline{Y}^{-1})^{R_1}$,
$$\begin{aligned}\Upsilon(\mathcal{B}) &\cong \langle R(R_1R_2)^3, R(R_2R_0)^3, R(\overline{X}\,\overline{Y}^{-1})^k\rangle^{\Delta/R} \\ &= \langle R(R_1R_2)^3, R(\overline{X}\,\overline{Y}^{-1})^k\rangle^{\Delta/R} \\ &= \langle R(R_1R_2)^3, R\overline{X}^k, R\overline{Y}^k\rangle^{\Delta/R} \\ &= \langle R(R_1R_2)^3, R[(R_1R_2)^3]^{R_0}, R[(R_1R_2)^3]^{R_0R_2}, R\overline{X}, R\overline{Y}\rangle^{\Delta/R} \\ &= \langle R(R_1R_2)^3, R\overline{X}^2, R\overline{Y}^2, R\overline{X}^k, R\overline{Y}^k\rangle^{\Delta/R} \\ &= \langle R(R_1R_2)^3, R\overline{X}^{\gcd(2,k)}, R\overline{Y}^{\gcd(2,k)}\rangle^{\Delta/R} \\ &= \langle R(R_1R_2)^3, R\overline{X}^{\gcd(2,k)}, R\overline{Y}^{\gcd(2,k)}\rangle.\end{aligned}$$
When $2 \nmid k$,
$$\begin{aligned}\Upsilon(\mathcal{B}) &\cong \langle R(R_1R_2)^3, R\overline{X}, R\overline{Y}\rangle \\ &= \langle R(R_1R_2)^3, R\overline{X}, R\overline{X}\,\overline{Y}\rangle \\ &\cong (C_{3k} \times C_k) \rtimes C_2,\end{aligned}$$
$\iota = 6k^2$ and $\mathcal{B}^\Delta$ is $\text{D}_{(02)}(\mathcal{P}_3) = \text{Walsh}(\mathcal{D}_3)$.
When $2 \mid k$,
$$\begin{aligned}\Upsilon(\mathcal{B}) &\cong \langle R(R_1R_2)^3, R\overline{X}^2, R\overline{Y}^2\rangle \\ &= \langle R(R_1R_2)^3, R\overline{X}^2, R\overline{X}^2\,\overline{Y}^2 = R(\overline{X}\,\overline{Y})^2\rangle \\ &\cong (C_{3k/2} \times C_{k/2}) \rtimes C_2,\end{aligned}$$

$\iota = 3k^2/2$ and $\mathcal{B}^\Delta$ is $\mathcal{C} = \mathrm{Walsh}(\mathrm{D}_{(12)}(\mathcal{T}))$.

- Case **13**: $\mathcal{B} = \mathrm{Walsh}(\mathrm{D}_{(12)}((4,2,4)_M))$, $d_1 = \gcd(l,m) = 4$.
  If $M = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, then $S = \{(R_0R_1R_2R_1)^k\overline{(12)}\} = \{(R_0R_2R_1R_2)^k\}$,
  $R(R_0R_2R_1R_2)^k\alpha_W = R(R_1R_2R_0R_2)^k = R[(R_0R_2R_1R_2)^k]^{R_2R_1} = R$,
  $\Upsilon(\mathcal{B}) \cong \langle R(R_1R_2)^4, R(R_2R_0)^4\rangle^{\Delta/R} = 1$ and $\iota = 1$.
  If $M = \begin{pmatrix} k & -k \\ k & k \end{pmatrix}$, then $S = \{(R_0R_1R_2)^{2k}\overline{(12)}\} = \{(R_0R_2R_1)^{2k}\}$,
  $R(R_0R_2R_1)^{2k}\alpha_W = R$,
  $\Upsilon(\mathcal{B}) \cong \langle R(R_1R_2)^4, R(R_2R_0)^4\rangle^{\Delta/R} = 1$ and $\iota = 1$.
  Either way, $\mathcal{B}$ is regular and $\mathcal{B}^\Delta = \mathcal{B}$.

Chirality groups and chirality indices of $\mathcal{B} = \mathrm{Pin}(\mathcal{R})$

The bipartite-regular hypermaps on the torus obtained by the Pin construction are listed, up to duality, in cases 1-6 of Table 4.1. We use the notations of Proposition 1.9.6.

Remark 4.3.5. Let $\mathcal{B} = \mathrm{Pin}(\mathcal{R})$ and ι the chirality index of $\mathcal{B}$. If $\mathcal{K}$ is a hypermap covered by $\mathcal{H}$ such that $\mathrm{Pin}(\mathcal{K})$ is regular and has $|\Omega_{\mathcal{B}}|/\iota$ flags, then $\mathcal{B}^\Delta = \mathrm{Pin}(\mathcal{K})$.

Note also that when $\mathcal{B} = \mathrm{Pin}(\mathcal{R})$, $\mathcal{B}^\Delta$ is a regular hypermap such that all vertices have valency 1 and hence, by Lemma 1.4.4, is on the sphere.

In order to facilitate our work we note that if $\{i,j,k\} = \{0,1,2\}$, then $(R_iR_jR_k)^2\alpha_P = 1$: if $j = 0$, then $(R_iR_jR_k)^2\alpha_P = ({R_1}^{R_0})^2 = 1$, else, if $j \neq 0$, then $(R_iR_jR_k)^2\alpha_P = (R_1)^2 = 1$.

- Case **1**: $\mathcal{B} = \mathrm{Pin}(\mathrm{D}_{(021)}((6,2,3)_M))$, $d_2 = \gcd(m,n) = 3$.
  If $M = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, then $S = \{(R_0R_1R_2)^{2k}\overline{(021)}\} = \{(R_2R_0R_1)^{2k}\}$, $R(R_2R_0R_1)^{2k}\alpha_P = R$,
  $$\begin{aligned}\Upsilon(\mathcal{B}) &\cong \langle RR_1R_2, R(R_0R_1)^3\rangle^{\Delta/R} = \langle RR_1R_2, R(R_1R_2)^{R_0}, R(R_0R_1)^3\rangle \\ &= \langle RR_1R_2, R(R_1R_2)^{R_0}R_1R_2, R(R_1R_2)^{R_0}(R_0R_1)^3\rangle \\ &\cong (C_k \times C_k) \rtimes C_2,\end{aligned}$$
  $\iota = 2k^2$ and $\mathcal{B}^\Delta$ is $\mathcal{S}_6 = \mathrm{Pin}(\mathcal{S}_3)$.
  If $M = \begin{pmatrix} k & -2k \\ k & k \end{pmatrix}$, then $S = \{(R_0R_1R_2R_1R_2)^{2k}\overline{(021)} = (R_2R_0R_1R_0R_1)^{2k}\}$,
  $R(R_2R_0R_1R_0R_1)^{2k}\alpha_P = R(R_0R_1R_0R_1R_0)^{2k} = R$,
  $$\begin{aligned}\Upsilon(\mathcal{B}) &\cong \langle RR_1R_2, R(R_0R_1)^3\rangle^{\Delta/R} = \langle RR_1R_2, R(R_1R_2)^{R_0}, R(R_0R_1)^3\rangle \\ &= \langle RR_1R_2, R(R_1R_2)^{R_0}R_1R_2, R(R_1R_2)^{R_0}(R_0R_1)^3\rangle \\ &\cong (C_{3k} \times C_k) \rtimes C_2,\end{aligned}$$
  $\iota = 6k^2$ and $\mathcal{B}^\Delta$ is $\mathcal{S}_6 = \mathrm{Pin}(\mathcal{S}_3)$.

- Case **2**: $\mathcal{B} = \mathrm{Pin}(\mathrm{D}_{(01)}((4,2,4)_M))$, $d_2 = \gcd(m,n) = 4$.
  Let $\overline{X} = X\overline{(01)}$ and $\overline{Y} = Y\overline{(01)}$. We have $R\overline{X} = RR_1R_0R_2R_0 = R(R_1R_2)(R_2R_0)^2$, $R\overline{Y} = RR_0R_1R_0R_2 = R(R_0R_1)^2(R_1R_2)$, $R\overline{X}\,\overline{Y} = R(R_0R_1)^2(R_2R_0)^2 = RR(R_1R_2)^{R_0R_1}(R_1R_2)$ and $R\overline{X}^{-1}\overline{Y} = R(R_1R_2)^{R_0}(R_1R_2)$.

If $M = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, then $S = \{(R_0R_1R_2R_1)^k\overline{(01)}\} = \{(R_1R_0R_2R_0)^k\}$, $R(R_1R_0R_2R_0)^k\alpha_P = R(R_0R_1R_0R_1)^k = R(R_0R_1)^{2k}$ and

$$\Upsilon(\mathcal{B}) \cong \langle RR_1R_2, R(R_0R_1)^4, R(R_0R_1)^{2k}\rangle^{\Delta/R} = \langle RR_1R_2, R(R_0R_1)^{\gcd(4,2k)}\rangle^{\Delta/R}.$$

When $2 \nmid k$, $R(R_2R_0)^2 = R(R_1R_2)^{R_0}(R_0R_1)^2(R_1R_2)$, so

$$\begin{aligned}\Upsilon(\mathcal{B}) &\cong \langle RR_1R_2, R(R_0R_1)^2\rangle^{\Delta/R} = \langle RR_1R_2, R(R_0R_1)^2\rangle^{\Delta/R} \\ &= \langle RR_1R_2, R(R_1R_2)^{R_0}, R(R_0R_1)^2\rangle \\ &= \langle RR_1R_2, R(R_0R_1)^2, R(R_2R_0)^2\rangle \\ &= \langle RR_1R_2, R\overline{X}, R\overline{Y}\rangle \\ &\cong (C_k \times C_k) \rtimes C_2,\end{aligned}$$

$\iota = 2k^2$ and $\mathcal{B}^\Delta$ is $\mathcal{S}_4 = \mathrm{Pin}(\mathcal{S}_2)$.
When $2 \mid k$,

$$\begin{aligned}\Upsilon(\mathcal{B}) &\cong \langle RR_1R_2, R(R_0R_1)^4\rangle^{\Delta/R} = \langle RR_1R_2\rangle^{\Delta/R} \\ &= \langle RR_1R_2, R(R_1R_2)^{R_0}, R(R_1R_2)^{R_0R_1}\rangle \\ &= \langle RR_1R_2, R\overline{X}\,\overline{Y}, R\overline{X}^{-1}\overline{Y}\rangle \\ &= \langle RR_1R_2, R\overline{X}^2, R\overline{X}\,\overline{Y}\rangle \\ &\cong (C_k \times C_{k/2}) \rtimes C_2,\end{aligned}$$

$\iota = k^2$ and $\mathcal{B}^\Delta$ is $\mathcal{S}_8 = \mathrm{Pin}(\mathcal{S}_4)$.
If $M = \begin{pmatrix} k & -k \\ k & k \end{pmatrix}$, then $S = \{(R_2R_1R_0)^{2k}\overline{(01)}\} = \{(R_2R_0R_1)^{2k}\}$, $R(R_2R_0R_1)^{2k}\alpha_P = R$,

$$\begin{aligned}\Upsilon(\mathcal{B}) &\cong \langle RR_1R_2, R(R_0R_1)^4\rangle^{\Delta/R} = \langle RR_1R_2\rangle^{\Delta/R} \\ &= \langle RR_1R_2, R(R_1R_2)^{R_0}, R(R_1R_2)^{R_0R_1}\rangle \\ &= \langle RR_1R_2, R\overline{X}\,\overline{Y}, R\overline{X}^{-1}\overline{Y}\rangle \\ &\cong (C_k \times C_k) \rtimes C_2,\end{aligned}$$

$\iota = 2k^2$ and $\mathcal{B}^\Delta$ is $\mathcal{S}_8 = \mathrm{Pin}(\mathcal{S}_4)$.

- Case **3**: $\mathcal{B} = \mathrm{Pin}(\mathrm{D}_{(02)}((6,2,3)_M))$, $d_2 = \gcd(m,n) = 2$.
  Let $\overline{X} = X\overline{(02)}$ and $\overline{Y} = Y\overline{(02)}$. We have $R\overline{X} = RR_2R_1R_0R_1R_0R_1 = R(R_1R_2)^{-1}(R_0R_1)^2$, $R\overline{Y} = RR_1R_2R_1R_0R_1R_0 = R(R_1R_2)^2[(R_1R_2)^{R_0}]^{-1}$.
  If $M = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, then $S = \{(R_0R_1R_2)^{2k}\overline{(02)}\} = \{(R_2R_1R_0)^{2k}\}$, $R(R_2R_1R_0)^{2k}\alpha_P = R$,

$$\begin{aligned}\Upsilon(\mathcal{B}) &\cong \langle RR_1R_2, R(R_0R_1)^2\rangle^{\Delta/R} \\ &= \langle RR_1R_2, R(R_1R_2)^{R_0}, R(R_0R_1)^2\rangle \\ &= \langle RR_1R_2, R\overline{X}, R\overline{Y}\rangle \\ &\cong (C_k \times C_k) \rtimes C_3,\end{aligned}$$

$\iota = 3k^2$ and $\mathcal{B}^\Delta$ is $\mathcal{S}_4 = \mathrm{Pin}(\mathcal{S}_2)$.
If $M = \begin{pmatrix} k & -2k \\ k & k \end{pmatrix}$, then $S = \{(R_0R_1R_2R_1R_2)^{2k}\overline{(02)}\} = \{(R_2R_1R_0R_1R_0)^{2k}\}$,

$R(R_2R_1R_0R_1R_0)^{2k}\alpha_P = R(R_0R_1R_0R_1R_0)^{2k} = R$,

$$\begin{aligned}\Upsilon(\mathcal{B}) &\cong \langle RR_1R_2, R(R_0R_1)^2\rangle^{\Delta/R} \\ &= \langle RR_1R_2, R(R_1R_2)^{R_0}, R(R_0R_1)^2\rangle \\ &= \langle RR_1R_2, R\overline{X}, R\overline{Y}\rangle \\ &= \langle RR_1R_2, R\overline{X}, R\overline{X}\ \overline{Y}\rangle \\ &\cong (C_{3k}\times C_k)\rtimes C_3,\end{aligned}$$

$\iota = 9k^2$ and $\mathcal{B}^\Delta$ is $\mathcal{S}_4 = \mathrm{Pin}(\mathcal{S}_2)$.

- Case **4**: $\mathcal{B} = \mathrm{Pin}((3,3,3)_M)$, $d_2 = \gcd(m,n) = 3$.
If $M = \begin{pmatrix} k & 0 \\ 0 & k\end{pmatrix}$, then $S = \{(R_2R_1R_0R_1)^k\}$,
$R(R_2R_1R_0R_1)^k\alpha_P = R(R_0R_0R_1R_0)^k = R(R_0R_1)^k$ and

$$\Upsilon(\mathcal{B}) \cong \langle RR_1R_2, R(R_0R_1)^3, R(R_0R_1)^k\rangle^{\Delta/R} = \langle RR_1R_2, R(R_0R_1)^{\gcd(3,k)}\rangle^{\Delta/R}.$$

We have $RXY = R[(R_1R_2)^{R_0}(R_1R_2)]^{-1}$ and $RX^{-2}Y = R(R_1R_2)[(R_1R_2)^{R_0}]^{-1}(R_1R_2)$.
When $3\nmid k$, $\Upsilon(\mathcal{B}) \cong \langle RR_1R_2, RR_0R_1\rangle^{\Delta/R} = \Delta^+/R \cong \mathrm{Aut}^+(\mathcal{R})$, $\iota = |\Omega_{\mathcal{R}}|/2 = 6k^2/2 = 3k^2$ and $\mathcal{B}^\Delta$ is $\mathcal{S}_2 = \mathrm{Pin}(\mathcal{S}_1)$.
When $3 \mid k$,

$$\begin{aligned}\Upsilon(\mathcal{B}) &\cong \langle RR_1R_2\rangle^{\Delta/R} \\ &= \langle RR_1R_2, R(R_1R_2)^{R_0}\rangle \\ &= \langle RR_1R_2, RXY, RX^{-2}Y\rangle \\ &= \langle RR_1R_2, RXY, RX^3\rangle \\ &\cong (C_k\times C_{k/3})\rtimes C_3,\end{aligned}$$

$\iota = k^2$ and $\mathcal{B}^\Delta$ is $\mathcal{S}_6 = \mathrm{Pin}(\mathcal{S}_3)$.
If $M = \begin{pmatrix} k & -2k \\ k & k\end{pmatrix}$, then $S = \{(R_2R_1R_0)^{2k}\}$, $R(R_2R_1R_0)^{2k}\alpha_P = R$,

$$\begin{aligned}\Upsilon(\mathcal{B}) &\cong \langle RR_1R_2, R(R_0R_1)^3\rangle^{\Delta/R} \\ &= \langle RR_1R_2\rangle^{\Delta/R} \\ &= \langle RR_1R_2, R(R_1R_2)^{R_0}\rangle \\ &= \langle RR_1R_2, RXY, RX^{-2}Y\rangle \\ &\cong (C_k\times C_k)\rtimes C_3,\end{aligned}$$

$\iota = 3k^2$ and $\mathcal{B}^\Delta$ is $\mathcal{S}_6 = \mathrm{Pin}(\mathcal{S}_3)$.

- Case **5**: $\mathcal{B} = \mathrm{Pin}((4,2,4)_M)$, $d_2 = \gcd(m,n) = 2$.
If $M = \begin{pmatrix} k & 0 \\ 0 & k\end{pmatrix}$, then $S = \{(R_0R_1R_2R_1)^k\}$,
$R(R_0R_1R_2R_1)^k\alpha_P = R(R_1R_0R_0R_0)^k = R(R_1R_0)^k$ and

$$\Upsilon(\mathcal{B}) \cong \langle RR_1R_2, R(R_0R_1)^2, R(R_1R_0)^k\rangle^{\Delta/R} = \langle RR_1R_2, R(R_0R_1)^{\gcd(2,k)}\rangle^{\Delta/R}.$$

When $2 \nmid k$, $\Upsilon(\mathcal{B}) \cong \langle RR_1R_2, RR_0R_1\rangle^{\Delta/R} = \Delta^+/R \cong \mathrm{Aut}^+(\mathcal{R})$, $\iota = |\Omega_{\mathcal{R}}|/2 = 8k^2/2 = 4k^2$ and $\mathcal{B}^{\Delta}$ is $\mathcal{S}_2 = \mathrm{Pin}(\mathcal{S}_1)$.
When $2 \mid k$, $RXY = R(R_1R_2)^{R_0}(R_1R_2)$, $RX^{-1}Y = R(R_1R_2)(R_0R_1)^{-2}(R_1R_2)$,

$$\begin{aligned}\Upsilon(\mathcal{B}) &\cong \langle RR_1R_2, R(R_0R_1)^2\rangle^{\Delta/R}\\ &= \langle RR_1R_2, R(R_1R_2)^{R_0}, R(R_0R_1)^2\rangle\\ &= \langle RR_1R_2, RXY, RX^{-1}Y\rangle\\ &= \langle RR_1R_2, RXY, RX^2\rangle\\ &= (C_k \times C_{k/2}) \times C_4,\end{aligned}$$

$\iota = 2k^2$ and $\mathcal{B}^{\Delta}$ is $\mathcal{S}_4 = \mathrm{Pin}(\mathcal{S}_2)$.
If $M = \begin{pmatrix} k & -k \\ k & k\end{pmatrix}$, then $S = \{(R_0R_1R_2)^{2k}\}$, $R(R_0R_1R_2)^{2k}\alpha_P = R$,

$$\begin{aligned}\Upsilon(\mathcal{B}) &\cong \langle RR_1R_2, R(R_0R_1)^2\rangle^{\Delta/R}\\ &= \langle RR_1R_2, R(R_1R_2)^{R_0}, R(R_0R_1)^2\rangle\\ &= \langle RR_1R_2, RXY, RX^{-1}Y\rangle\\ &= (C_k \times C_k) \times C_4,\end{aligned}$$

$\iota = 4k^2$ and $\mathcal{B}^{\Delta}$ is $\mathcal{S}_4 = \mathrm{Pin}(\mathcal{S}_2)$.

- Case **6**: $\mathcal{B} = \mathrm{Pin}((6,2,3)_M)$, $d_2 = \gcd(m,n) = 1$. From Corollary 1.9.7, we see that $\Upsilon(\mathcal{B}) \cong \Delta^+/R \cong \mathrm{Aut}^+(\mathcal{R})$.
If $M = \begin{pmatrix} k & 0 \\ 0 & k\end{pmatrix}$, then $\iota = |\Omega_{\mathcal{R}}|/2 = 12k^2/2 = 6k^2$.
If $M = \begin{pmatrix} k & -2k \\ k & k\end{pmatrix}$, then $\iota = |\Omega_{\mathcal{R}}|/2 = 36k^2/2 = 18k^2$.
In both cases $\mathcal{B}^{\Delta}$ is $\mathcal{S}_2 = \mathrm{Pin}(\mathcal{S}_1)$.

4.4 A note on restrictedly-regular hypermaps on the Klein bottle

In [33], Coxeter and Moser show that there are no regular maps on the Klein bottle, and in [15], Breda and Jones extend this result to hypermaps. However, the Klein bottle has Θ-regular hypermaps for every $\Theta \lhd_2 \Delta$, $\Theta \neq \Delta^+$. The hypermap $\mathcal{B}$ with hypermap subgroup $B = \langle (R_1R_2)^4, (R_2R_0)^2, (R_0R_1)^4\rangle^{\Delta}\langle {X_4}^2, Y_4, X_4Y_4R_2\rangle$ is a $\Delta^{\hat{1}}$-regular hypermap on the Klein bottle with 8 flags, 1 vertex, 2 edges and 1 face. It is obtained from its orientable double covering $\mathcal{B}^+ = (4,2,4)_{\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1\end{smallmatrix}\right)}$ and the involutory $\Delta^{\hat{1}}$-conservative orientation-reversing automorphism of $\mathcal{B}^+$ which maps B^+ to $B^+X_4Y_4R_2$. Similarly, the hypermap $\mathcal{P}$ with hypermap subgroup $P = \langle (R_1R_2)^4, (R_2R_0)^2, (R_0R_1)^4\rangle^{\Delta}\langle {X_4}^2{Y_4}^2, {X_4}^{-1}Y_4, X_4Y_4R_1\rangle$ is a Δ^1-regular hypermap on the Klein bottle with 16 flags, 2 vertices, 4 edges and 2 faces. It is obtained from its orientable double covering $\mathcal{P}^+ = (4,2,4)_{\left(\begin{smallmatrix} 2 & -1 \\ 2 & 1\end{smallmatrix}\right)}$ and the involutory Δ^1-conservative orientation-reversing automorphism of $\mathcal{P}^+$ which maps P^+ to $P^+X_4Y_4R_1$. Other 2-restrictedly-regular hypermaps on the Klein bottle can be obtained by duality.

We intend to classify the 2-restrictedly-regular hypermaps on the Klein bottle in a future work.

#	$\mathcal{U}$	Note	$\mathcal{U}^{\Delta}$	Υ	ι
1	$(4,2,4)_{\left(\begin{smallmatrix} l & -m \\ m & l \end{smallmatrix}\right)}$	$2k \mid l-m$	$(4,2,4)_{\left(\begin{smallmatrix} k & -k \\ k & k \end{smallmatrix}\right)}$	$C_{(l^2+m^2)/2k^2}$	$\frac{l^2+m^2}{2k^2}$
		$2k \nmid l-m$	$(4,2,4)_{\left(\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}\right)}$	$C_{(l^2+m^2)/k^2}$	$\frac{l^2+m^2}{k^2}$
2	$(6,2,3)_{\left(\begin{smallmatrix} l & -l-m \\ m & l \end{smallmatrix}\right)}$	$3k \mid l-m$	$(6,2,3)_{\left(\begin{smallmatrix} k & -2k \\ k & k \end{smallmatrix}\right)}$	$C_{(l^2+lm+m^2)/3k^2}$	$\frac{l^2+lm+m^2}{3k^2}$
		$3k \nmid l-m$	$(6,2,3)_{\left(\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}\right)}$	$C_{(l^2+lm+m^2)/k^2}$	$\frac{l^2+lm+m^2}{k^2}$
3	$(3,3,3)_{\left(\begin{smallmatrix} l & -l-m \\ m & l \end{smallmatrix}\right)}$	$3k \mid l-m$	$(3,3,3)_{\left(\begin{smallmatrix} k & -2k \\ k & k \end{smallmatrix}\right)}$	$C_{(l^2+lm+m^2)/3k^2}$	$\frac{l^2+lm+m^2}{3k^2}$
		$3k \nmid l-m$	$(3,3,3)_{\left(\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}\right)}$	$C_{(l^2+lm+m^2)/k^2}$	$\frac{l^2+lm+m^2}{k^2}$
4	$(4,2,4)_{\left(\begin{smallmatrix} l & -m \\ l & m \end{smallmatrix}\right)}$		$(4,2,4)_{\left(\begin{smallmatrix} k & -k \\ k & k \end{smallmatrix}\right)}$	C_{lm/k^2}	$\frac{lm}{k^2}$
5	$(4,2,4)_{\left(\begin{smallmatrix} l & m \\ m & l \end{smallmatrix}\right)}$	$2k \mid l-m$	$(4,2,4)_{\left(\begin{smallmatrix} k & -k \\ k & k \end{smallmatrix}\right)}$	$C_{\lvert l^2-m^2\rvert/2k^2}$	$\frac{\lvert l^2-m^2\rvert}{2k^2}$
		$2k \nmid l-m$	$(4,2,4)_{\left(\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}\right)}$	$C_{\lvert l^2-m^2\rvert/k^2}$	$\frac{\lvert l^2-m^2\rvert}{k^2}$
6	$(4,2,4)_{\left(\begin{smallmatrix} l & 0 \\ 0 & m \end{smallmatrix}\right)}$		$(4,2,4)_{\left(\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}\right)}$	C_{lm/k^2}	$\frac{lm}{k^2}$
7	$(4,2,4)_{\left(\begin{smallmatrix} l & -l \\ m & m \end{smallmatrix}\right)}$	$2k \mid l-m$	$(4,2,4)_{\left(\begin{smallmatrix} k & -k \\ k & k \end{smallmatrix}\right)}$	C_{lm/k^2}	$\frac{lm}{k^2}$
		$2k \nmid l-m$	$(4,2,4)_{\left(\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}\right)}$	C_{2lm/k^2}	$\frac{2lm}{k^2}$

Table 4.2: Chirality groups, chirality indices and closure covers of the 2-restrictedly-regular uniform hypermaps on the torus. ($k = \gcd(l, m)$)

#	$\mathcal{B}$	Note	$\mathcal{B}^{\triangle}$	Υ	ι
1	$\mathrm{Pin}(\mathrm{D}_{(021)}((6,2,3)_M))$	$M = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$	$\mathcal{S}_6$	$(C_k \times C_k) \rtimes C_2$	$2k^2$
		$M = \begin{pmatrix} k & -2k \\ k & k \end{pmatrix}$	$\mathcal{S}_6$	$(C_{3k} \times C_k) \rtimes C_2$	$6k^2$
2	$\mathrm{Pin}(\mathrm{D}_{(01)}((4,2,4)_M))$	$M = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, 2 \nmid k$	$\mathcal{S}_4$	$(C_k \times C_k) \rtimes C_2$	$2k^2$
		$M = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, 2 \mid k$	$\mathcal{S}_8$	$(C_k \times C_{k/2}) \rtimes C_2$	k^2
		$M = \begin{pmatrix} k & -k \\ k & k \end{pmatrix}$	$\mathcal{S}_8$	$(C_k \times C_k) \rtimes C_2$	$2k^2$
3	$\mathrm{Pin}(\mathrm{D}_{(02)}((6,2,3)_M))$	$M = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$	$\mathcal{S}_4$	$(C_k \times C_k) \rtimes C_3$	$3k^2$
		$M = \begin{pmatrix} k & -2k \\ k & k \end{pmatrix}$	$\mathcal{S}_4$	$(C_{3k} \times C_k) \rtimes C_3$	$9k^2$
4	$\mathrm{Pin}((3,3,3)_M)$	$M = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, 3 \nmid k$	$\mathcal{S}_2$	$(C_k \times C_k) \rtimes C_3$	$3k^2$
		$M = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, 3 \mid k$	$\mathcal{S}_6$	$(C_k \times C_{k/3}) \rtimes C_3$	k^2
		$M = \begin{pmatrix} k & -2k \\ k & k \end{pmatrix}$	$\mathcal{S}_6$	$(C_k \times C_k) \rtimes C_3$	$3k^2$
5	$\mathrm{Pin}((4,2,4)_M)$	$M = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, 2 \nmid k$	$\mathcal{S}_2$	$(C_k \times C_k) \rtimes C_4$	$4k^2$
		$M = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, 2 \mid k$	$\mathcal{S}_4$	$(C_k \times C_{k/2}) \times C_4$	$2k^2$
		$M = \begin{pmatrix} k & -k \\ k & k \end{pmatrix}$	$\mathcal{S}_4$	$(C_k \times C_k) \times C_4$	$4k^2$
6	$\mathrm{Pin}((6,2,3)_M)$	$M = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$	$\mathcal{S}_2$	$(C_k \times C_k) \rtimes C_6$	$6k^2$
		$M = \begin{pmatrix} k & -2k \\ k & k \end{pmatrix}$	$\mathcal{S}_2$	$(C_{3k} \times C_k) \rtimes C_6$	$18k^2$
7	$\mathrm{Walsh}(\mathrm{D}_{(02)}((6,2,3)_M))$	$M = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$	$\mathcal{S}_2$	$(C_k \times C_k) \rtimes C_6$	$6k^2$
		$M = \begin{pmatrix} k & -2k \\ k & k \end{pmatrix}$	$\mathcal{S}_2$	$(C_{3k} \times C_k) \rtimes C_6$	$18k^2$
8	$\mathrm{Walsh}((4,2,4)_M)$	$M = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, 2 \nmid k$	$\mathcal{S}_2$	$(C_k \times C_k) \rtimes C_4$	$4k^2$
		$M = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, 2 \mid k$	$\mathcal{P}_8$	$(C_{k/2} \times C_{k/2}) \rtimes C_2$	$k^2/2$
		$M = \begin{pmatrix} k & -k \\ k & k \end{pmatrix}, 2 \nmid k$	$\mathcal{P}_4$	$(C_k \times C_k) \rtimes C_2$	$2k^2$
		$M = \begin{pmatrix} k & -k \\ k & k \end{pmatrix}, 2 \mid k$	$\mathcal{P}_8$	$(C_k \times C_{k/2}) \rtimes C_2$	k^2
9	$\mathrm{Walsh}((6,2,3)_M)$	$M = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, 3 \nmid k$	$\mathcal{P}_2$	$(C_k \times C_k) \rtimes C_3$	$3k^2$
		$M = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, 3 \mid k$	$\mathcal{P}_6$	$(C_k \times C_{k/3}) \rtimes C_3$	k^2
		$M = \begin{pmatrix} k & -2k \\ k & k \end{pmatrix}$	$\mathcal{P}_6$	$(C_k \times C_k) \rtimes C_3$	$3k^2$
10	$\mathrm{Walsh}((3,3,3)_M)$		$\mathcal{B}$	1	1
11	$\mathrm{Walsh}(\mathrm{D}_{(12)}((6,2,3)_M))$	$M = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, 2 \nmid k$	$\mathrm{D}_{(02)}(\mathcal{P}_3)$	$(C_k \times C_k) \rtimes C_2$	$2k^2$
		$M = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, 2 \mid k$	$\mathcal{C}$	$(C_{k/2} \times C_{k/2}) \rtimes C_2$	$k^2/2$
		$M = \begin{pmatrix} k & -2k \\ k & k \end{pmatrix}, 2 \nmid k$	$\mathrm{D}_{(02)}(\mathcal{D}_3)$	$(C_{3k} \times C_k) \rtimes C_2$	$6k^2$
		$M = \begin{pmatrix} k & -2k \\ k & k \end{pmatrix}, 2 \mid k$	$\mathcal{C}$	$(C_{3k/2} \times C_{k/2}) \rtimes C_2$	$3k^2/2$
12	$\mathrm{Walsh}(\mathrm{D}_{(12)}((4,2,4)_M))$	$M = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$	$\mathcal{B}$	1	1

Table 4.3: Chirality groups, chirality indices and closure covers of the bipartite-regular hypermaps on the torus obtained via the Walsh and Pin constructions.

Chapter 5

Hypermaps on the double torus

In this chapter we deal with the 2-restrictedly-regular hypermaps on the double torus.

The classification of the orientably-regular maps on the double torus was made by Threlfall [62] in 1932, completing earlier work of Brahana [3] (see Table 9 of [33]). In 1988, Corn and Singerman [28] determined all possible types for the remaining orientably-regular hypermaps on the double torus (those of type (l, m, n) with $l, m, n \geq 3$), as well as their rotation groups (see Table 2 of [28]). Breda and Jones [15] classified the orientably-regular hypermaps on the double torus and computed their rotation and automorphism groups. More recently, Singerman and Syddall [59] determined the number of isomorphism classes of uniform hypermaps on the double torus using using Conder's small index subgroup programme [20].

5.1 Regular and orientably-regular hypermaps on the double torus

Because the double torus is an orientable surface, the regular hypermaps on the double torus are among the orientably-regular.

In this section we assume that $\mathcal{H}$ is an orientably-regular hypermap on the double torus of type (l, m, n). Since $\mathcal{H}$ is uniform and has characteristic -2, using the Euler formula for uniform hypermaps (Corollary 1.4.2), we get

$$\frac{|\Omega_{\mathcal{H}}|}{2} = \frac{2}{1 - (1/l + 1/m + 1/n)}. \tag{5.1}$$

Naturally, l, m, n divide $|\Omega_{\mathcal{H}}|/2$ because $V = |\Omega_{\mathcal{H}}|/2l$, $E = |\Omega_{\mathcal{H}}|/2m$ and $F = |\Omega_{\mathcal{H}}|/2n$ are the numbers of vertices, edges and faces of $\mathcal{H}$, respectively. In addition, Theorem 1.4.6 states that $|\Omega_{\mathcal{H}}| \leq -84\chi = 168$, or equivalently, $|\Omega_{\mathcal{H}}|/2 \leq 84$.

Table 5.1 lists all 22 possibilities for the number $|\Omega|$ of flags and type (l, m, n) of a uniform hypermap on the double torus, with $l \leq m \leq n$ as well as its numbers of vertices, edges and faces. These values were obtained using GAP [34]. In the last column we display the number of orientably-regular hypermaps on the double torus of type (l, m, n) (determined by Breda and Jones in [15]). The numbers of non-isomorphic uniform hypermaps on the double torus of type (l, m, n), with $l \leq m \leq n$, can be found in [59].

In what follows we give a brief description of how to find all orientably-regular hypermaps on the double torus.

#	$\lvert\Omega\rvert$	l	m	n	V	E	F	orient.-reg.
1	10	5	5	5	1	1	1	3
2	12	3	6	6	2	1	1	1
3	16	2	8	8	4	1	1	1
4	16	4	4	4	2	2	2	1
5	18	3	3	9	3	3	1	0
6	20	2	5	10	5	2	1	1
7	24	2	4	12	6	3	1	0
8	24	2	6	6	6	2	2	1
9	24	3	3	6	4	4	2	0
10	24	3	4	4	4	3	3	1
11	30	3	3	5	5	5	3	0
12	32	2	4	8	8	4	2	1
13	36	2	3	18	9	6	1	0
14	40	2	5	5	10	4	4	0
15	48	2	3	12	12	8	2	0
16	48	2	4	6	12	6	4	1
17	48	3	3	4	8	8	6	1
18	60	2	3	10	15	10	3	0
19	72	2	3	9	18	12	4	0
20	80	2	4	5	20	10	8	0
21	96	2	3	8	24	16	6	1
22	168	2	3	7	42	28	12	0

Table 5.1: All possible values for the number of flags and type of an orientably-regular hypermap on the double torus

Let H be a hypermap subgroup of $\mathcal{H}$, $G^+ := \Delta^+/H$, $x := HR_1R_2$, $y := HR_2R_0$ and $z := HR_0R_1$. Then $G^+ = \langle x, y, z\rangle$ and $xyz = 1$.

Using the Sylow theorems it is easy to show that there are no orientably-regular hypermaps corresponding to cases 11, 14, 20 and 22. In cases 20 and 22, n is prime and $\langle z\rangle$ is the unique n-Sylow-subgroup. Hence $\langle z\rangle \lhd G^+$. Since $xy = z^{-1} \in \langle z\rangle$, $\langle z\rangle x = \langle z\rangle y^{-1}$. It follows that $\langle z\rangle = \langle z\rangle x^2 = \langle z\rangle y^{-2}$ and $y^{-2} \in \langle z\rangle$. On the other hand, Lagrange's theorem ensures that $y^{-2} \notin \langle z\rangle$ because in both cases the order of y^{-2} does not divide the order of z. In cases 11 and 14, m is prime and $\langle y\rangle$ is the unique m-Sylow-subgroup. In case 11, x has order m, so $x \in \langle y\rangle$ but $z = (xy)^{-1} \notin \langle y\rangle$ because $n \nmid m$. Similarly, in case 14, z has order m, so $z \in \langle y\rangle$ but $x = (yz)^{-1} \notin \langle y\rangle$ because $l \nmid m$.

A brief consideration shows that there are no orientably-regular hypermaps corresponding to cases 9 and 15. First of all, we remark that if a face is adjacent to itself, then it is unique. Indeed, if Hg and HgR_2 are flags on the same face f, then $HgR_2R_0 = Hgx$ is also in f. It follows that $x \in \langle z\rangle$, so $\langle z\rangle = \langle x, z\rangle = G^+$. Second, y (resp. x) induces a permutation of the faces incident at an edge (resp. a vertex) such that all its disjoint cycles have the same length. Clearly, this length divides the valency m (resp. l) of all edges (resp. vertices). Similarly, z induces a permutation of the faces adjacent to a face such that all its disjoint cycles have the same length d. This length d is divides n and must be smaller than n. Finally, we note that a hypermap corresponding to cases 9 or 15 has 2 faces, and its edges have valency 3.

In cases 1, 2, 3, 5, 6, 7 and 13, $F = 1$, so $|G^+| = |\langle z \rangle|$, that is $G^+ = \langle z \rangle \cong C_n$. It follows that there is $0 \leq k < n$ such that $x = z^k$ and $y = x^{-1}z^{-1} = z^{-k-1}$. Because x and y have orders l and m, $\gcd(n, k) = n/l$ and $\gcd(n, k+1) = n/m$.

In cases 4, 8 and 12 (as well as in cases 9 and 15), $F = 2$, so $|G^+| = 2|\langle z \rangle|$ and hence $\langle z \rangle \lhd_2 G^+$. Since $x^2, y^2 \in \langle z \rangle \lhd_2 G^+$, there are $0 \leq j, k < n$ such that $x^2 = z^j$ and $y^2 = z^k$. In addition $l/\gcd(l, 2) = n/\gcd(n, j)$ and $m/\gcd(m, 2) = n/\gcd(n, j)$. Alternatively, note that $z^x, z^y \in \langle z \rangle$ because $\langle z \rangle \lhd_2 G^+$, so there are $0 \leq p, q < n$ such that $z^x = z^p$, $z^y = z^q$ and $\gcd(n, p) = 1 = \gcd(n, q)$.

In case 10, the number of faces incident at each edge must be 2, so $y^2 \in \langle z \rangle$. Because y and z have order 4, $y^2 = z^2$.

In cases 18 and 19, the number d of faces adjacent to a face is 2 and 3, respectively. Then $(z^d)^x \in \langle z \rangle$, that is, $(z^d)^x = z^k$ for some $0 \leq k < n$.

In case 21, the number d of faces adjacent to a face is 2 or 4. Either way $(z^4)^x \in \langle z \rangle$. Because z has order 8, z^4 and $(z^4)^x$ have order 2, so $(z^4)^x = z^4$, that is, $(z^4x)^2 = 1$.

In case 16, the number d of faces adjacent to a face is 2 or 3. Because each face is adjacent to the same number of faces, d cannot be 3. Then $d = 2$ and $(y^2)^x \in \langle y \rangle$. Having in mind that y has order 4 and x has order 2, $(y^2)^x = y^2 = y^{-2}$, that is, $(y^2x)^2 = 1$.

In case 17, z induces 2 permutations of the faces adjacent to a face such that their order divides 4. The disjoint cycles of these 2 permutations must have the same length, so they have order 1 or 2. Either way $(z^2)^x \in \langle z \rangle$ and, because z has order 4, $(z^2)^x = z^2$.

With the help of GAP [34], this last procedure allows us to find hypermap subgroups for the orientably-regular hypermaps on the double torus. In each case we can determine a finite set T, contained in H and containing $S = \{(R_1R_2)^l, (R_2R_0)^m, (R_0R_1)^n\}$, such that $[\Delta^+ : \langle T \rangle^{\Delta^+}] = |\Omega_{\mathcal{H}}|/2$. Clearly, $\mathcal{H}$ is regular if and only if $H = \langle T \rangle^{\Delta}$, or equivalently, if and only if $[\Delta : \langle T \rangle^{\Delta}] = 2[\Delta^+ : \langle T \rangle^{\Delta}] = 2[\Delta^+ : H]$. By inspection, or using GAP [34] again, we get:

Theorem 5.1.1 (Breda and Jones [15]). *All orientably-regular hypermaps on the double torus are regular.*

In other words, there are no orientably-chiral hypermaps on the double torus.

Table 5.2 lists, up to duality, all regular hypermaps on the double torus. For each regular hypermap $\mathcal{R}$ on the double torus of type (l, m, n) with $l \leq m \leq n$ we give a list X of additional relations such that the normal closure in Δ of $T := S \cup X$ is a hypermap subgroup of $\mathcal{R}$. Finally, in the last two columns we give the rotation group, $\mathrm{Aut}^+(\mathcal{R})$ and the automorphism group, $\mathrm{Aut}(\mathcal{R})$, which can be found in [15]. In the semi-direct product $C_3 \rtimes C_4$, the generator of C_4 acts on C_3 by inverting its elements. This group is denoted by $\langle 2, 3, 3 \rangle$ in [33] and by $\widehat{D}_3$ in [15]. Notice that the hypermaps in lines 1, 2 and 3 are not isomorphic. However, $\mathcal{H}_2 \cong \mathrm{D}_{(12)}(\mathcal{H}_1)$, $\mathcal{H}_3 \cong \mathrm{D}_{(01)}(\mathcal{H}_1)$ and $\mathcal{H}_1 \cong \mathrm{D}_{(02)}(\mathcal{H}_1)$. The automorphism group of the hypermap $\mathcal{H}_{13}$ is the group of genus two [63], the unique group for which the minimum genus over all surfaces containing an imbedded Cayley graph for the group is two.

Lemma 5.1.2 (Conservativeness of the regular hypermaps on the double torus). *Let $\Theta \lhd_2 \Delta$ and let $\mathcal{H}_j$ be the regular hypermap listed in line j of Table 5.2. Then:*

1. *$\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ and $\mathcal{H}_{12}$ are Θ-conservative if and only if $\Theta = \Delta^+$;*
2. *$\mathcal{H}_4$, $\mathcal{H}_5$ and $\mathcal{H}_9$ are Θ-conservative if and only if $\Theta \in \{\Delta^+, \Delta^0, \Delta^{\hat{0}}\}$;*

#	$\mathcal{R}$	$\|\Omega\|$	l	m	n	Additional relations	$\mathrm{Aut}^+(\mathcal{R})$	$\mathrm{Aut}(\mathcal{R})$	Θ-cons.
1	$\mathcal{H}_1$	10	5	5	5	$(R_1R_2)(R_0R_1)^{-1}$	C_5	D_5	Δ^+
2	$\mathcal{H}_2$	10	5	5	5	$(R_1R_2)(R_0R_1)^{-2}$	C_5	D_5	Δ^+
3	$\mathcal{H}_3$	10	5	5	5	$(R_1R_2)(R_0R_1)^{-3}$	C_5	D_5	Δ^+
4	$\mathcal{H}_4$	12	3	6	6	$(R_1R_2)(R_0R_1)^{-4}$	C_6	D_6	$\Delta^+, \Delta^0, \Delta^{\hat{0}}$
5	$\mathcal{H}_5$	16	2	8	8	$(R_1R_2)(R_0R_1)^{-4}$	C_8	D_8	$\Delta^+, \Delta^0, \Delta^{\hat{0}}$
6	$\mathcal{H}_6$	16	4	4	4	$(R_1R_2)^2(R_0R_1)^{-2}$, $(R_2R_0)^2(R_0R_1)^{-2}$	Q_8	$Q_8 \cdot C_4$	all
7	$\mathcal{H}_7$	20	2	5	10	$(R_1R_2)(R_0R_1)^{-5}$	C_{10}	D_{10}	$\Delta^+, \Delta^1, \Delta^{\hat{1}}$
8	$\mathcal{H}_8$	24	2	6	6	$(R_2R_0)^2(R_0R_1)^{-4}$	$C_6 \times C_2$	$D_6 \times C_2$	all
9	$\mathcal{H}_9$	24	3	4	4	$(R_2R_0)^2(R_0R_1)^{-2}$	$C_3 \rtimes C_4$	$(4,6 \mid 2,2)$	$\Delta^+, \Delta^0, \Delta^{\hat{0}}$
10	$\mathcal{H}_{10}$	32	2	4	8	$(R_2R_0)^2(R_0R_1)^{-4}$	$\langle -2,4 \mid 2\rangle$	$\mathrm{Hol}(C_8)$	all
11	$\mathcal{H}_{11}$	48	2	4	6	$((R_2R_0)^2(R_1R_2))^2$	$(4,6 \mid 2,2)$	$D_3 \times D_4$	all
12	$\mathcal{H}_{12}$	48	3	3	4	$[(R_0R_1)^2, (R_1R_2)]$	$SL_2(3)$	$GL_2(3)$	Δ^+
13	$\mathcal{H}_{13}$	96	2	3	8	$((R_0R_1)^4(R_1R_2))^2$	$GL_2(3)$	$GL_2(3) \rtimes C_2$	$\Delta^+, \Delta^1, \Delta^{\hat{1}}$

Table 5.2: The orientably-regular hypermaps on the double torus 'up to duality'

3. $\mathcal{H}_7$ and $\mathcal{H}_{13}$ are Θ-conservative if and only if $\Theta \in \{\Delta^+, \Delta^1, \Delta^{\hat{1}}\}$;

4. $\mathcal{H}_6$, $\mathcal{H}_8$, $\mathcal{H}_{10}$ and $\mathcal{H}_{11}$ are Θ-conservative for all $\Theta \lhd_2 \Delta$.

5.2 Pseudo-orientably-regular and bipartite-regular hypermaps on the double torus

Since $\Delta^{+0\hat{0}} = \Delta^+ \cap \Delta^0 = \Delta^+ \cap \Delta^{\hat{0}}$, every pseudo-orientably-regular hypermap $\mathcal{P}$ on an orientable surface $\mathcal{S}$ is $\Delta^{+0\hat{0}}$-regular, as well as every bipartite-regular hypermap $\mathcal{B}$ on $\mathcal{S}$. For this reason, we can derive the classifications of pseudo-orientably-regular and bipartite-regular hypermaps on $\mathcal{S}$ from the classification of $\Delta^{+0\hat{0}}$-regular hypermaps on $\mathcal{S}$.

In this section we determine all $\Delta^{+0\hat{0}}$-regular hypermaps on the double torus in order to classify all pseudo-orientably-regular and bipartite-regular hypermaps on the double torus.

Now we assume that $\mathcal{H}$ is a $\Delta^{+0\hat{0}}$-regular hypermap and H is a hypermap subgroup of $\mathcal{H}$. According to Lemma 1.3.9, $\mathcal{H}$ is bipartite-uniform. Let $(l_1, l_2; m; n)$ be the bipartite-type of $\mathcal{H}$. Since $H \lhd \Delta^{+0\hat{0}}$ and $|\Omega_{\mathcal{H}}| = [\Delta : H] = [\Delta : \Delta^{+0\hat{0}}] \cdot [\Delta^{+0\hat{0}} : H] = 4[\Delta^{+0\hat{0}} : H]$, $\Delta^{+0\hat{0}}/H$ is a group with order $|\Omega_{\mathcal{H}}|/4$. By the Euler formula for bipartite-uniform hypermaps (Corollary 1.4.3), $(a, b, c, d) = (l_1, l_2, m/2, n/2)$ is a solution of

$$\frac{|\Omega_{\mathcal{H}}|}{4} = \frac{2}{2 - (1/a + 1/b + 1/c + 1/d)}, \tag{5.2}$$

such that $a, b, c, d \mid |\Omega_{\mathcal{H}}|/4$. Theorem 1.4.8 states that $|\Omega_{\mathcal{H}}| \leq -168\chi = 336$, or equivalently, $|\Omega_{\mathcal{H}}|/4 \leq 84$.

Using GAP [34], one can easily determine all values for $|\Omega_{\mathcal{H}}|/4$ and (a, b, c, d) such that $a \leq b \leq c \leq d$, $|\Omega|/4$ is a multiple of a, b, c, d and equation (5.2) holds. These values are listed in Table 5.3 and give rise a total of 119 distinct values for the bipartite-type $(l_1, l_2; m; n)$ of a bipartite-uniform hypermap on the double torus, with $l_1 \leq l_2$ and $m \leq n$.

#	$\|\Omega\|$	a	b	c	d	#	$\|\Omega\|$	a	b	c	d
1	12	3	3	3	3	**15**	48	1	3	4	4
2	16	2	2	4	4	**16**	48	2	2	2	3
3	20	1	5	5	5	**17**	60	1	3	3	5
4	24	1	3	6	6	**18**	64	1	2	4	8
5	24	2	2	2	6	**19**	72	1	2	3	18
6	24	2	2	3	3	**20**	80	1	2	5	5
7	32	1	2	8	8	**21**	96	1	2	3	12
8	32	1	4	4	4	**22**	96	1	2	4	6
9	32	2	2	2	4	**23**	96	1	3	3	4
10	36	1	3	3	9	**24**	120	1	2	3	10
11	40	1	2	5	10	**25**	144	1	2	3	9
12	48	1	2	4	12	**26**	160	1	2	4	5
13	48	1	2	6	6	**27**	192	1	2	3	8
14	48	1	3	3	6	**28**	336	1	2	3	7

Table 5.3: Solutions of (5.2) with $a \leq b \leq c \leq d$

If $l_1 = 1$ (53 cases) or $m/2 = 1$ (another 53 cases), then $\mathcal{H} \cong \mathrm{Pin}(\mathcal{K})$ or $\mathcal{H} \cong \mathrm{Walsh}(\mathcal{K})$ for some hypermap $\mathcal{K}$ on the double torus. Since $\Delta^{+0\hat{0}} = \Delta^+ \cap \Delta^{\hat{0}} = \Delta^+ {\varphi_P}^{-1} = \Delta^+ {\varphi_W}^{-1}$ and $\mathcal{H}$ is $\Delta^{+0\hat{0}}$-regular, $\mathcal{K}$ is Δ^+-regular, that is, orientably-regular. By Theorem 5.1.1, $\mathcal{K}$ is regular and hence $\mathcal{H}$ is bipartite-regular. This shows the following result.

Lemma 5.2.1. *If $\mathcal{H}$ is a $\Delta^{+0\hat{0}}$-regular hypermap on the double torus obtained by the* Pin *or* Walsh *construction, then $\mathcal{H}$ is bipartite-regular. Thus, $\mathcal{H}$ is Δ^0-regular if and only if $\mathcal{H}$ is regular.*

The bipartite-regular hypermaps on the double torus obtained by the Pin and Walsh constructions are displayed in Tables 5.5 and 5.4, respectively. Because $\mathcal{H}_3 \cong \mathrm{D}_{(01)}(\mathcal{H}_1)$, $\mathrm{Walsh}(\mathcal{H}_3) \cong \mathrm{Walsh}(\mathrm{D}_{(01)}(\mathcal{H}_1)) \cong \mathrm{Walsh}(\mathcal{H}_1)$ (by Theorem 1.6.6). Since $\mathcal{H}_2 \cong \mathrm{D}_{(12)}(\mathcal{H}_1)$, $\mathrm{Pin}(\mathcal{H}_2) \cong \mathrm{Pin}(\mathrm{D}_{(12)}(\mathcal{H}_1)) \cong \mathrm{D}_{(12)}(\mathrm{Pin}(\mathcal{H}_1))$ (by Theorem 1.6.10); however $\mathrm{Pin}(\mathcal{H}_2)$ is not isomorphic to $\mathrm{Pin}(\mathcal{H}_1)$.

Remark 5.2.2. The bipartite-regular hypermaps on the double torus obtained by the Pin construction are non-uniform and hence bipartite-chiral.

The Walsh construction gives rise to 17 non-uniform bipartite-regular hypermaps on the double torus. One easily checks that

- $\mathrm{Walsh}(\mathcal{H}_2) = \mathrm{D}_{(01)}(\mathcal{H}_7)$,
- $\mathrm{Walsh}(\mathrm{D}_{(02)}(\mathcal{H}_4)) = \mathrm{D}_{(01)}(\mathcal{H}_8)$,
- $\mathrm{Walsh}(\mathrm{D}_{(02)}(\mathcal{H}_5)) = \mathrm{D}_{(012)}(\mathcal{H}_{10})$,
- $\mathrm{Walsh}(\mathcal{H}_6) = \mathrm{D}_{(01)}(\mathcal{H}_{10})$,
- $\mathrm{Walsh}(\mathrm{D}_{(02)}(\mathcal{H}_8)) = \mathrm{D}_{(012)}(\mathcal{H}_{11})$,
- $\mathrm{Walsh}(\mathrm{D}_{(02)}(\mathcal{H}_9)) = \mathcal{H}_{11}$,

#	$\|\Omega\|$	l_1	l_2	$m/2$	$n/2$	Walsh$(\cdot)$
1	20	5	5	1	5	$\mathcal{H}_2$
2	20	5	5	1	5	$\mathcal{H}_1$ or $\mathcal{H}_3$
3	24	3	6	1	6	$\mathcal{H}_4$
4	24	6	6	1	3	$\mathrm{D}_{(02)}(\mathcal{H}_4)$
5	32	2	8	1	8	$\mathcal{H}_5$
6	32	8	8	1	2	$\mathrm{D}_{(02)}(\mathcal{H}_5)$
7	32	4	4	1	4	$\mathcal{H}_6$
8	40	2	5	1	10	$\mathcal{H}_7$
9	40	2	10	1	5	$\mathrm{D}_{(12)}(\mathcal{H}_7)$
10	40	5	10	1	2	$\mathrm{D}_{(021)}(\mathcal{H}_7)$
11	48	2	6	1	6	$\mathcal{H}_8$
12	48	6	6	1	2	$\mathrm{D}_{(02)}(\mathcal{H}_8)$

#	$\|\Omega\|$	l_1	l_2	$m/2$	$n/2$	Walsh$(\cdot)$
13	48	3	4	1	4	$\mathcal{H}_9$
14	48	4	4	1	3	$\mathrm{D}_{(02)}(\mathcal{H}_9)$
15	64	2	4	1	8	$\mathcal{H}_{10}$
16	64	2	8	1	4	$\mathrm{D}_{(12)}(\mathcal{H}_{10})$
17	64	4	8	1	2	$\mathrm{D}_{(021)}(\mathcal{H}_{10})$
18	96	2	4	1	6	$\mathcal{H}_{11}$
19	96	2	6	1	4	$\mathrm{D}_{(12)}(\mathcal{H}_{11})$
20	96	4	6	1	2	$\mathrm{D}_{(021)}(\mathcal{H}_{11})$
21	96	3	3	1	4	$\mathcal{H}_{12}$
22	96	3	4	1	3	$\mathrm{D}_{(12)}(\mathcal{H}_{12})$
23	192	2	3	1	8	$\mathcal{H}_{13}$
24	192	2	8	1	3	$\mathrm{D}_{(12)}(\mathcal{H}_{13})$
25	192	3	8	1	2	$\mathrm{D}_{(021)}(\mathcal{H}_{13})$

Table 5.4: Bipartite-regular hypermaps on the double torus obtained by the Walsh construction.

- Walsh$(\mathcal{H}_{12}) = \mathcal{H}_{13}$,
- Walsh$(\mathcal{H}_1) \cong$ Walsh$(\mathcal{H}_3)$ is bipartite-regular and uniform but it is not regular.

In other words, the hypermaps listed in lines 1, 4, 6, 7, 12, 14 and 21 of Table 5.4 are regular, so their chirality groups are trivial and their chirality indices are 1; the other 18 hypermaps are bipartite-chiral.

It remains to analyze the $\Delta^{+0\hat{0}}$-regular hypermaps which are not obtained via the Walsh and Pin constructions. The list of all possible values for the bipartite-type $(l_1, l_2; m; n)$ of a $\Delta^{+0\hat{0}}$-regular hypermap on the double torus which is not constructed by the Walsh or Pin constructions (that is, such that $l_1, l_2, m/2, n/2 \geq 2$) is given in Table 5.6.

Let $\mathcal{H}$ be one of these $\Delta^{+0\hat{0}}$-regular hypermaps, H a hypermap subgroup of $\mathcal{H}$ and $G^* = \Delta^{+0\hat{0}}/H$. Let $r = HR_1R_2$, $s = H(R_1R_2)^{R_0}$, $t = H(R_2R_0)^2 = HR_2{R_2}^{R_0}$ and $u = H(R_0R_1)^2 = H{R_1}^{R_0}R_1$. Since $\Delta^{+0\hat{0}} = \langle R_1R_2, (R_1R_2)^{R_0}, (R_2R_0)^2, (R_0R_1)^2\rangle$ (see Section 1.2), $G^*/H = \langle r, s, t, u\rangle$ and $s = urt$. In addition:

- $r^{R_1} = r^{R_2} = r^{-1}$, $r^{R_0} = s = urt$;
- $t^{R_2} = t^{R_0} = t^{-1}$, $t^{R_1} = (t^{-1})^{r^{-1}}$;
- $u^{R_0} = u^{R_1} = u^{-1}$, $u^{R_2} = (u^{-1})^{r}$;
- $s^{R_0} = r$, $s^{R_1} = (urt)^{R_1} = u^{-1}r^{-1}(t^{-1})^{r^{-1}} = u^{-1}t^{-1}r^{-1}$, $s^{R_2} = (s^{R_1})^r = r^{-1}u^{-1}t^{-1}$.

In case 1, $G^* = \langle u\rangle \cong C_3$. Because s has order 3, u, r, t cannot be all equal. So $u = r^{-1} = t = s$ or $u = r = t^{-1} = s$ or $u^{-1} = r = t = s$. The relations $u = r = t^{-1} = s$ or $u^{-1} = r = t = s$ give rise to hypermaps subgroups which are conjugate under R_0. For this reason there are 2 non-isomorphic $\Delta^{+0\hat{0}}$-regular hypermaps of type $(3, 3; 6; 6)$.

#	$\lvert\Omega\rvert$	l_1	l_2	$m/2$	$n/2$	Pin(·)
1	20	1	5	5	5	$\mathcal{H}_1$
2	20	1	5	5	5	$\mathcal{H}_2$
3	20	1	5	5	5	$\mathcal{H}_3$
4	24	1	3	6	6	$\mathcal{H}_4$
5	24	1	6	3	6	$\mathrm{D}_{(01)}(\mathcal{H}_4)$
6	32	1	2	8	8	$\mathcal{H}_5$
7	32	1	8	2	8	$\mathrm{D}_{(01)}(\mathcal{H}_5)$
8	32	1	4	4	4	$\mathcal{H}_6$
9	40	1	2	5	10	$\mathcal{H}_7$
10	40	1	5	2	10	$\mathrm{D}_{(01)}(\mathcal{H}_7)$
11	40	1	10	2	5	$\mathrm{D}_{(012)}(\mathcal{H}_7)$
12	48	1	2	6	6	$\mathcal{H}_8$
13	48	1	6	2	6	$\mathrm{D}_{(01)}(\mathcal{H}_8)$
14	48	1	3	4	4	$\mathcal{H}_9$
15	48	1	4	3	4	$\mathrm{D}_{(01)}(\mathcal{H}_9)$
16	64	1	2	4	8	$\mathcal{H}_{10}$
17	64	1	4	2	8	$\mathrm{D}_{(01)}(\mathcal{H}_{10})$
18	64	1	8	2	4	$\mathrm{D}_{(012)}(\mathcal{H}_{10})$
19	96	1	2	4	6	$\mathcal{H}_{11}$
20	96	1	4	2	6	$\mathrm{D}_{(01)}(\mathcal{H}_{11})$
21	96	1	6	2	4	$\mathrm{D}_{(012)}(\mathcal{H}_{11})$
22	96	1	3	3	4	$\mathcal{H}_{12}$
23	96	1	4	3	3	$\mathrm{D}_{(02)}(\mathcal{H}_{12})$
24	192	1	2	3	8	$\mathcal{H}_{13}$
25	192	1	3	2	8	$\mathrm{D}_{(01)}(\mathcal{H}_{13})$
26	192	1	8	2	3	$\mathrm{D}_{(012)}(\mathcal{H}_{13})$

Table 5.5: Bipartite-regular hypermaps on the double torus obtained by the Pin construction.

#	$\lvert\Omega\rvert$	l_1	l_2	$m/2$	$n/2$
1	12	3	3	3	3
2	16	2	2	4	4
3	16	2	4	2	4
4	16	4	4	2	2
5	24	2	2	2	6
6	24	2	6	2	2
7	24	2	2	3	3
8	24	2	3	2	3
9	24	3	3	2	2
10	32	2	2	2	4
11	32	2	4	2	2
12	48	2	2	2	3
13	48	2	3	2	2

Table 5.6: Possible values for the bipartite-type $(l_1, l_2; m; n)$ of a $\Delta^{+0\hat{0}}$-regular hypermap on the double torus which is not obtained from regular hypermaps by the Walsh or the Pin constructions.

In case 2, $G^* = \langle u \rangle \cong C_4$. This group only has 1 element of order 2, so $r = s = u^2$. In addition, since G^* is abelian, $s = urt$ implies that $t = u^{-1}$. In case 3, $G^* = \langle u \rangle \cong C_4$, $r = t = u^2$ and $s = u$. In case 4, $G^* = \langle r \rangle \cong C_4$, $t = u = r^2$ and $s = r$. For each bipartite-type $(2, 2; 8; 8)$, $(2, 4; 4; 8)$ and $(4, 4; 4; 4)$ there is only one $\Delta^{+0\hat{0}}$-regular hypermap with such type.

In cases 5 and 6, G^* has elements of order 6, so $G^* \cong C_6$ is abelian. However, $s^2 \neq u^2r^2t^2$ because exactly one of these elements does not have order 2. Consequently there is no $\Delta^{+0\hat{0}}$ hypermap corresponding to these cases.

In cases 7, 8 and 9, G^* is C_6 or $D_3 \cong S_3$ and G^* has exactly one 3-Sylow-subgroup. In case 7, if $G^* = C_6$, then $u \leftrightharpoons r$ and $s = r$, because G^* is abelian and has only 1 element of order 2; else, if $G^* \cong D_3$, then $u \neq u^r = u^{-1}$, so $t = u$ and $r = s$, or $t = u^{-1}$ and $s = ru$. These last relations give rise to hypermaps subgroups which are conjugate under R_1. In case 8, if $G^* = C_6$, then $u \leftrightharpoons r$ and $t = r$; else, if $G^* \cong D_3$, then $u \neq u^r = u^{-1}$, so $s = u$ and $r = t$, or $s = u^{-1}$ and $t = ru$. In case 9, if $G^* = C_6$, then $u \leftrightharpoons r$ and $t = u$; else, if $G^* \cong D_3$, then $r \neq r^u = r^{-1}$, so $s = r$ and $u = tr$, or $s = r^{-1}$ and $t = u$. These last relations give rise to hypermaps subgroups which are conjugate under R_1. Because of this, there are 2 non-isomorphic $\Delta^{+0\hat{0}}$-regular hypermaps of bipartite-type $(2, 2; 6; 6)$, 3 non-isomorphic $\Delta^{+0\hat{0}}$-regular hypermaps of bipartite-type $(2, 3; 4; 6)$ and 2 non-isomorphic $\Delta^{+0\hat{0}}$-regular hypermaps of bipartite-type $(3, 3; 4; 4)$.

In cases 10 and 11, G^* is not abelian because $s^2 \neq u^2r^2t^2$. In addition $G^* \neq Q_8$, for otherwise the unique element of Q_8 that has order 2 is in the center of Q_8, so $s = urt$ implies $r = s = t = u$. Hence $G^* = D_4$. In case 10, r, s, t cannot be all in $\langle u \rangle \lhd_2 G^*$. On the other hand $s = urt$ implies that at the number of elements of $\{r, s, t\}$ outside $\langle u \rangle$ is even. If $r, s \notin \langle u \rangle$, then $u^r = u^{-1}$, $t = u^2$ and $s = ru$; if $r, t \notin \langle u \rangle$, then $u^r = u^{-1}$, $s = u^2$ and $t = ru$; if $s, t \notin \langle u \rangle$, then $u^t = u^{-1}$, $r = u^2$ and $s = tu$. These last 2 sets of relations give rise to hypermaps subgroups which are conjugate under R_0. In case 11, $\langle s \rangle \lhd_2 G^*$; if $r, t \notin \langle s \rangle$, then $u = s^2$ and $t = sr$; if $r, u \notin \langle s \rangle$, then $t = s^2$ and $u = rs$; if $t, u \notin \langle s \rangle$, then $r = s^2$ and $u = ts$. These first 2 sets of relations give rise to hypermaps subgroups, H and $H\overline{(12)}$, of non-isomorphic dual hypermaps. Because of this, there are 2 non-isomorphic $\Delta^{+0\hat{0}}$-regular hypermaps of bipartite-type $(2, 2; 4; 8)$ and 3 non-isomorphic $\Delta^{+0\hat{0}}$-regular hypermaps of bipartite-type $(2, 4; 4; 4)$.

Finally, in cases 12 and 13, since $s^2 \neq u^2r^2t^2$, G^* is not abelian. There are 3 non-abelian groups of order 12: $D_6 \cong D_3 \times C_2$, A_4 and $C_3 \rtimes C_4$. The number of 3-Sylow-subgroups of a group with 12 elements is 1 ou 4; if the number of 3-Sylow-subgroups if 4, then there are 8 elements of order 3 and the remaining 4 elements form the only 2-Sylow-subgroup of G^*. However, because every involution is in a 2-Sylow-subgroup and $s = urt$, or $u^{-1} = rts^{-1}$, both groups cannot have just 1 2-Sylow-subgroup. Hence $G^* \neq A_4$. In addition, G^* cannot be $C_3 \rtimes C_4 = \langle R, S \mid S^3 = T^2 = (ST)^2 \rangle$ because this group has exactly 1 element of order 2 which generates the center of the group, so if 3 elements of $\{r, s, t, u\}$ have order 2, then $s = urt$ implies that all have order 2 and $r = s = t = u$. Indeed $C_3 \rtimes C_4$ has 1 element of order 1, 1 of order 2, 2 of order 3, 6 of order 4 and 2 of order 6. Reasoning by elimination we get $G^* \cong D_6$. Let $N \cong C_6 \lhd G^*$. In case 12, $u \in N \lhd_2 G^*$. Because $s = urt$, the set $\{r, s, t\}$ has 1 element inside and 2 outside G^*. If $r, s \notin N$, then $t, u \in N$ and $t \leftrightharpoons u$; if $s, t \notin N$, then $r, u \in N$ and $r \leftrightharpoons u$; if $r, t \notin N$, then $s, u \in N$ and $s \leftrightharpoons u$. These last 2 sets of relations give rise to hypermaps subgroups which are conjugate under R_0. In case 13, $s \in N \lhd_2 G^*$. If $r, t \notin N$, then $u, s \in N$ and $u \leftrightharpoons s$; if $r, u \notin N$, then $t, s \in N$ and $t \leftrightharpoons s$; if $t, u \notin N$, then $r, s \in N$ and $r \leftrightharpoons s$. These first 2 sets of relations give rise to hypermaps subgroups, H and

#	$\|\Omega\|$	a,b,c,d	Additional relations	$\mathrm{N}_\Delta(H)$	ι	G^*
1	12	$3,3,3,3$	$(R_0R_1)^2[(R_2R_0)^2]^{-1}, (R_0R_1)^2(R_1R_2)$	Δ	1	C_3
2	12	$3,3,3,3$	$(R_0R_1)^2(R_2R_0)^2, (R_0R_1)^2(R_1R_2)^{-1}$	$\Delta^{\hat{0}}$	3	C_3
3	16	$2,2,4,4$	$(R_0R_1)^2(R_2R_0)^2, [(R_0R_1)^2]^2(R_1R_2)^{-1}$	Δ	1	C_4
4	16	$2,4,2,4$	$[(R_0R_1)^2]^2(R_1R_2)^{-1}, [(R_0R_1)^2]^2[(R_2R_0)^2]^{-1}$	$\Delta^{\hat{0}}$	4	C_4
5	16	$4,4,2,2$	$(R_1R_2)^2[(R_2R_0)^2]^{-1}, (R_1R_2)^2[(R_0R_1)^2]^{-1}$	Δ	1	C_4
6	24	$2,2,3,3$	$[(R_0R_1)^2]^{R_1R_2}[(R_0R_1)^2]^{-1}, ({R_1}^{R_0}{R_2}^{R_0})(R_1R_2)^{-1}$	Δ	1	C_6
7	24	$2,2,3,3$	$[(R_1R_2)(R_0R_1)^2]^2, (R_2R_0)^2[(R_0R_1)^2]^{-1}$	Δ^0	3	D_3
8	24	$2,3,2,3$	$[(R_0R_1)^2]^{R_1R_2}[(R_0R_1)^2]^{-1}, (R_2R_0)^2(R_1R_2)^{-1}$	$\Delta^{\hat{0}}$	6	C_6
9	24	$2,3,2,3$	$[(R_1R_2)(R_0R_1)^2]^2, ({R_1}^{R_0}{R_2}^{R_0})[(R_0R_1)^2]^{-1}$	$\Delta^{\hat{0}}$	6	D_3
10	24	$2,3,2,3$	$[(R_1R_2)(R_0R_1)^2]^2, ({R_1}^{R_0}{R_2}^{R_0})[(R_0R_1)^2]$	$\Delta^{\hat{0}}$	6	D_3
11	24	$3,3,2,2$	$[(R_0R_1)^2]^{R_1R_2}[(R_0R_1)^2]^{-1}, (R_2R_0)^2[(R_0R_1)^2]^{-1}$	Δ	1	C_6
12	24	$3,3,2,2$	$[(R_0R_1)^2(R_1R_2)]^2, ({R_1}^{R_0}{R_2}^{R_0})(R_1R_2)^{-1}$	Δ^0	3	D_3
13	32	$2,2,2,4$	$[(R_0R_1)^2]^2[(R_2R_0)^2]^{-1}, (R_1R_2)(R_0R_1)^2({R_1}^{R_0}{R_2}^{R_0})^{-1}$	Δ	1	D_4
14	32	$2,2,2,4$	$[(R_0R_1)^2]^2({R_1}^{R_0}{R_2}^{R_0})^{-1}, (R_1R_2)(R_0R_1)^2[(R_2R_0)^2]^{-1}$	$\Delta^{\hat{0}}$	4	D_4
15	32	$2,4,2,2$	$({R_1}^{R_0}{R_2}^{R_0})^2[(R_0R_1)^2]^{-1}, ({R_1}^{R_0}{R_2}^{R_0})(R_1R_2)(R_2R_0)^2$	$\Delta^{\hat{0}}$	2	D_4
16	32	$2,4,2,2$	$({R_1}^{R_0}{R_2}^{R_0})^2[(R_2R_0)^2]^{-1}, ({R_1}^{R_0}{R_2}^{R_0})(R_0R_1)^2(R_1R_2)$	$\Delta^{\hat{0}}$	2	D_4
17	32	$2,4,2,2$	$({R_1}^{R_0}{R_2}^{R_0})^2(R_1R_2)^{-1}, ({R_1}^{R_0}{R_2}^{R_0})(R_0R_1)^2(R_2R_0)^2$	$\Delta^{\hat{0}}$	4	D_4
18	48	$2,2,2,3$	$(R_2R_0)^2(R_0R_1)^2(R_2R_0)^2[(R_0R_1)^2]^{-1}$	Δ	1	D_6
19	48	$2,2,2,3$	$(R_1R_2)(R_0R_1)^2(R_1R_2)[(R_0R_1)^2]^{-1}$	$\Delta^{\hat{0}}$	3	D_6
20	48	$2,3,2,2$	$(R_0R_1)^2({R_1}^{R_0}{R_2}^{R_0})(R_0R_1)^2({R_1}^{R_0}{R_2}^{R_0})^{-1}$	$\Delta^{\hat{0}}$	6	D_6
21	48	$2,3,2,2$	$(R_2R_0)^2({R_1}^{R_0}{R_2}^{R_0})(R_2R_0)^2({R_1}^{R_0}{R_2}^{R_0})^{-1}$	$\Delta^{\hat{0}}$	6	D_6
22	48	$2,3,2,2$	$(R_1R_2)({R_1}^{R_0}{R_2}^{R_0})(R_1R_2)({R_1}^{R_0}{R_2}^{R_0})^{-1}$	$\Delta^{\hat{0}}$	6	D_6

Table 5.7: $\Delta^{+0\hat{0}}$-regular hypermaps on the double torus which are not obtained by the Pin or Walsh constructions

$\overline{H(12)}$, of non-isomorphic dual hypermaps. Because of this, there are 2 non-isomorphic $\Delta^{+0\hat{0}}$-regular hypermaps of bipartite-type $(2,2;4;6)$ and 3 non-isomorphic $\Delta^{+0\hat{0}}$-regular hypermaps of bipartite-type $(2,3;4;4)$.

Table 5.7 lists all $\Delta^{+0\hat{0}}$-regular hypermaps $\mathcal{H}$ on the double torus which are not obtained by the Walsh or Pin constructions. It also displays a list X of additional relations such that the normal closure in $\Delta^{+0\hat{0}}$ of $T := X \cup \{(R_1R_2)^a, [(R_1R_2)^{R_0}]^b, (R_2R_0)^{2c}, (R_0R_1)^{2d}\}$ is a hypermap subgroup H of $\mathcal{H}$.

Using GAP [34], one can determine if $\mathcal{H}$ is regular or 2-restrictedly-regular in the following way. The normalizer N in Δ of H, containing $\Delta^{+0\hat{0}}$, is $\Delta^{+0\hat{0}}$, Δ^+, Δ^0, $\Delta^{\hat{0}}$ or Δ. Theorem 5.1.1 states that every orientably-regular hypermap on the double torus is regular, so N cannot be Δ^+. Let $\Theta \in \{\Delta^0, \Delta^{\hat{0}}\}$. Now $N = \mathrm{N}_\Delta(H)$ contains Θ if and only if $H^\Theta = H$, or equivalently, if and only if

$$[\Theta : T^\Theta] = [\Theta : H^\Theta] = [\Theta : \Delta^{+0\hat{0}}] \cdot [\Delta^{+0\hat{0}} : H^\Theta] = 2 \cdot [\Delta^{+0\hat{0}} : H] = 2[\Delta^{+0\hat{0}} : T^{\Delta^{+0\hat{0}}}]. \quad (5.3)$$

Furthermore, if $H = T^\Theta$ is not normal in Λ, where $\{\Theta, \Lambda\} = \{\Delta^0, \Delta^{\hat{0}}\}$, then $H^\Delta = H^\Lambda$ and

$$|\Omega_{\mathcal{H}}| = [\Delta : H] = [\Delta : \Lambda] \cdot [\Lambda : H^\Delta] \cdot [H^\Delta : H] = 2 \cdot [\Lambda : H^\Lambda] \cdot [H^\Delta : H], \quad (5.4)$$

so the chirality index of $\mathcal{H}$ is equal to $|\Omega_{\mathcal{H}}|/(2[\Lambda : H^{\Lambda}])$. Obviously, when $\mathcal{H}$ is not uniform, we just need to check if $\mathcal{H}$ is bipartite-regular or not, since $\mathcal{H}$ cannot be Δ^0-regular or regular. In the last columns of Table 5.7 we display the normalizer in Δ of H, the chirality index ι of $\mathcal{H}$ and the group $G^* = \Delta^{+0\hat{0}}/H$.

Remark 5.2.3. The hypermaps listed in lines 1, 3, 5, 6, 11, 13 and 18 of Table 5.7 are the regular hypermaps $\mathcal{H}_4$, $\mathcal{H}_5$, $\mathcal{H}_6$, $\mathcal{H}_8$, $\mathcal{H}_9$, $\mathcal{H}_{10}$ and $\mathcal{H}_{11}$ of Table 5.2.

5.3 Chirality groups and chirality indices of the 2-restrictedly-regular hypermaps on the double torus

In this section we compute the chirality groups and chirality indices of the 2-restrictedly-regular hypermaps on the torus.

According to Theorem 5.1.1, there are no orientably-chiral hypermaps on the double torus. Looking at Tables 5.5, 5.4 and 5.7 and Remarks 5.2.2 and 5.2.3, we can see that, up to duality, there are 4 pseudo-orientably-chiral and 60 bipartite-chiral hypermaps on the double torus.

5.3.1 Chirality groups and chirality indices of the bipartite-regular hypermaps on the double torus obtained by the Walsh or Pin constructions

Chirality groups and chirality indices of $\mathcal{B} = \mathrm{Walsh}(\mathcal{R})$

Let $\mathcal{W}_j = \mathrm{Walsh}(\mathcal{O}_j)$ be the bipartite-regular hypermap on the double torus listed in line j of Table 5.4. Since $\mathcal{W}_j$ is $\Delta^{+0\hat{0}}$-regular, $\mathcal{W}_j$ covers $\mathcal{S}_2$. Let O_j be a hypermap subgroup of $\mathcal{O}_j$, $x := O_jR_1R_2$, $y := O_jR_2R_0$ and $z := O_jR_0R_1$. Then $\Delta^+/O_j = \langle x, y, z\rangle$ and $xyz = 1$.

- If j is 1, 4, 6, 7, 12, 14 or 21, then $\mathcal{W}_j$ is regular, so $\Upsilon(\mathcal{W}_j) = 1$ and $\mathcal{W}_j{}^{\Delta} = \mathcal{W}_j$.
- If j is 8, 13, 22, 23 or 25, then, by Corollary 1.9.7, $\Upsilon(\mathcal{W}_j) \cong \mathrm{Aut}^+(\mathcal{O}_j)$ and $\mathcal{W}_j{}^{\Delta} = \mathcal{S}_2$.
- If j is 9, 11, 15, 16, 17, 18 or 19, $\Upsilon(\mathcal{W}_j) \cong \langle y^2\rangle \cong C_{m/2}$ and $\iota = m/2$. Let $p = |\Omega|/4m$. In all 7 cases $\mathcal{O}_j \to \mathcal{P}_p$, so $\mathcal{W}_j = \mathrm{Walsh}(\mathcal{O}_j) \to \mathrm{Walsh}(\mathcal{P}_p) \cong \mathcal{P}_{2p}$ and $\mathcal{W}_j{}^{\Delta} = \mathcal{P}_{2p}$.
- If j is 2 or 5, $\Upsilon(\mathcal{W}_j) \cong \langle z\rangle \cong C_n$ and $\mathcal{W}_j{}^{\Delta} = \mathcal{S}_2$.
- If j is 3 or 10, then $\Upsilon(\mathcal{W}_j) \cong \langle y^l\rangle \cong C_{m/l}$ and $\mathcal{W}_j{}^{\Delta} \cong \mathrm{D}_{(02)}(\mathcal{P}_l)$. In both cases $\mathcal{O}_j \to \mathcal{D}_l$, so $\mathcal{W}_j = \mathrm{Walsh}(\mathcal{O}_j) \to \mathrm{Walsh}(\mathcal{D}_l) \cong \mathrm{D}_{(02)}(\mathcal{P}_l)$ and $\mathcal{W}_j{}^{\Delta} = \mathrm{D}_{(02)}(\mathcal{P}_l)$.
- $\Upsilon(\mathcal{W}_{20}) \cong \langle y^2, x^2\rangle \cong C_3 \times C_2 \cong C_6$ because $x^2 \in \mathrm{Z}(\Delta^+/O_{20})$. Since $\mathcal{O}_{20} \to \mathcal{P}_2$, $\mathcal{W}_{20} = \mathrm{Walsh}(\mathcal{O}_{20}) \to \mathrm{Walsh}(\mathcal{P}_2) \cong \mathcal{P}_4$ and $\mathcal{W}_{20}{}^{\Delta} = \mathcal{P}_4$.
- $\Upsilon(\mathcal{W}_{24}) \cong \langle y^2, (y^2)^z\rangle \cong Q_8$, since $y^4 \in \mathrm{Z}(\Delta^+/O_{24})$, $[(y^2)^z]^2 = (y^4)^z = y^4 = (y^2)^2$ and $(y^2(y^2)^z)^2 = (y^2x)^4 = [(yxy)^4]^{y^{-1}} = [(xyx)^{-4}]^{y^{-1}} = [xy^{-4}x]^{y^{-1}} = (y^4)^{xy^{-1}} = y^4$. Since $\mathcal{O}_{24} \to \mathcal{P}_3$, $\mathcal{W}_{24} = \mathrm{Walsh}(\mathcal{O}_{24}) \to \mathrm{Walsh}(\mathcal{P}_3) \cong \mathcal{P}_6$ and $\mathcal{W}_{24}{}^{\Delta} = \mathcal{P}_6$.

Table 5.8 lists the chirality groups, chirality indices and closure covers of the bipartite-regular hypermaps on the double torus obtained by the Walsh construction.

#	$\mathcal{B} = \text{Walsh}(\cdot)$	Υ	ι	$\mathcal{B}^{\Delta}$
1	$\text{Walsh}(\mathcal{H}_2)$	1	1	$\text{D}_{(01)}(\mathcal{H}_7)$
2	$\text{Walsh}(\mathcal{H}_1) \cong \text{Walsh}(\mathcal{H}_3)$	C_5	5	$\mathcal{S}_2$
3	$\text{Walsh}(\mathcal{H}_4)$	C_2	2	$\text{D}_{(02)}(\mathcal{P}_3)$
4	$\text{Walsh}(\text{D}_{(02)}(\mathcal{H}_4))$	1	1	$\text{D}_{(01)}(\mathcal{H}_8)$
5	$\text{Walsh}(\mathcal{H}_5)$	C_8	8	$\mathcal{S}_2$
6	$\text{Walsh}(\text{D}_{(02)}(\mathcal{H}_5))$	1	1	$\text{D}_{(012)}(\mathcal{H}_{10})$
7	$\text{Walsh}(\mathcal{H}_6)$	1	1	$\text{D}_{(01)}(\mathcal{H}_{10})$
8	$\text{Walsh}(\mathcal{H}_7)$	C_{10}	10	$\mathcal{S}_2$
9	$\text{Walsh}(\text{D}_{(12)}(\mathcal{H}_7))$	C_5	5	$\mathcal{P}_2$
10	$\text{Walsh}(\text{D}_{(021)}(\mathcal{H}_7))$	C_2	2	$\text{D}_{(02)}(\mathcal{P}_5)$
11	$\text{Walsh}(\mathcal{H}_8)$	C_3	3	$\mathcal{P}_4$
12	$\text{Walsh}(\text{D}_{(02)}(\mathcal{H}_8))$	1	1	$\text{D}_{(012)}(\mathcal{H}_{11})$
13	$\text{Walsh}(\mathcal{H}_9)$	$C_3 \rtimes C_4$	12	$\mathcal{S}_2$
14	$\text{Walsh}(\text{D}_{(02)}(\mathcal{H}_9))$	1	1	$\mathcal{H}_{11}$
15	$\text{Walsh}(\mathcal{H}_{10})$	C_2	2	$\mathcal{P}_8$
16	$\text{Walsh}(\text{D}_{(12)}(\mathcal{H}_{10}))$	C_4	4	$\mathcal{P}_4$
17	$\text{Walsh}(\text{D}_{(021)}(\mathcal{H}_{10}))$	C_4	4	$\mathcal{P}_4$
18	$\text{Walsh}(\mathcal{H}_{11})$	C_2	2	$\mathcal{P}_{12}$
19	$\text{Walsh}(\text{D}_{(12)}(\mathcal{H}_{11}))$	C_3	3	$\mathcal{P}_8$
20	$\text{Walsh}(\text{D}_{(021)}(\mathcal{H}_{11}))$	C_6	6	$\mathcal{P}_4$
21	$\text{Walsh}(\mathcal{H}_{12})$	1	1	$\mathcal{H}_{13}$
22	$\text{Walsh}(\text{D}_{(12)}(\mathcal{H}_{12}))$	$SL_2(3)$	24	$\mathcal{S}_2$
23	$\text{Walsh}(\mathcal{H}_{13})$	$GL_2(3)$	48	$\mathcal{S}_2$
24	$\text{Walsh}(\text{D}_{(12)}(\mathcal{H}_{13}))$	Q_8	8	$\mathcal{P}_6$
25	$\text{Walsh}(\text{D}_{(021)}(\mathcal{H}_{13}))$	$GL_2(3)$	48	$\mathcal{S}_2$

Table 5.8: Chirality groups, chirality indices and closure covers of the bipartite-regular hypermaps on the double torus obtained by the Walsh construction.

Chirality groups and chirality indices of $\mathcal{B} = \mathrm{Pin}(\mathcal{R})$

Let $\mathcal{P}_j = \mathrm{Pin}(\mathcal{O}_j)$ be the bipartite-regular hypermap on the double torus listed in line j of Table 5.5. Since $\mathcal{P}_j$ is $\Delta^{+0\hat{0}}$-regular, $\mathcal{P}_j$ covers $\mathcal{S}_2$. By Proposition 1.8.5, ${\mathcal{P}_j}^{\Delta}$ has type $(1, 2k, 2k)$ and hence ${\mathcal{P}_j}^{\Delta} = \mathcal{S}_{2k}$, for some $k \in \mathbb{N}$. Let O_j be a hypermap subgroup of $\mathcal{O}_j$, $x := O_j R_1 R_2$, $y := O_j R_2 R_0$ and $z := O_j R_0 R_1$. Then $\Delta^+/O_j = \langle x, y, z\rangle$ and $xyz = 1$.

- If j is 11, 15, 22, 24 or 26, then, by Corollary 1.9.7, $\Upsilon(\mathcal{P}_j) \cong \mathrm{Aut}^+(\mathcal{O}_j)$ and ${\mathcal{P}_j}^{\Delta} = \mathcal{S}_2$.
- By Corollary 1.9.7, $\Upsilon(\mathcal{P}_7) \cong \mathrm{Aut}^+(\mathrm{D}_{(01)}(\mathcal{H}_5)) \cong \mathrm{Aut}^+(\mathcal{H}_5)$ and ${\mathcal{P}_7}^{\Delta} = \mathcal{S}_2$, because $d_2 = 2$ but $\mathcal{O}_7 = \mathrm{D}_{(01)}(\mathcal{H}_5)$ is not bipartite (see Lemma 5.1.2).
- If j is 10, 13, 17, 18, 19, 20, 21 or 25, then by Corollary 1.9.7, $\Upsilon(\mathcal{P}_j) \cong \mathrm{Aut}^{+0\hat{0}}(\mathcal{O}_j)$ and ${\mathcal{P}_j}^{\Delta} = \mathcal{S}_4$, because $d_2 = 2$ and $\mathcal{O}_j$ is bipartite (see Lemma 5.1.2).
  Since $\mathrm{Aut}^{+0\hat{0}}(\mathcal{O}_j)$ is a subgroup of index 2 in $\mathrm{Aut}^+(\mathcal{O}_j)$, and C_{10} and $GL_2(3)$ just have one subgroup of index 2, $\Upsilon(\mathcal{P}_{10}) \cong C_5$ and $\Upsilon(\mathcal{P}_{25}) \cong SL_2(3)$.
  In case 13, $\Upsilon(\mathcal{P}_{13}) \cong C_6$ because all 3 subgroups of $C_6 \times C_2 \cong V_4 \times C_3$ of index 2 are isomorphic to C_6.
  In case 17, $x^2 = z^4 = (z^2)^2$ and $z^x = x^2 xzx = z^3(zx)^2 = z^3 y^{-2} = z^3$; therefore $(xz^2)^2 = x^2 x^{-1} z^2 x z^2 = x^2 z^6 z^2 = x^2$ and $\Upsilon(\mathcal{P}_{17}) \cong \langle x, z^2\rangle \cong Q_8$.
  In case 18, $\Upsilon(\mathcal{P}_{18}) \cong \langle x\rangle \cong C_8$ because x has order 8.
  In case 19, $(y^2x)^2 = 1$ implies that $(z^2)^x = z^{-2}$ and that $x \leftrightharpoons y^2$. Since $y \leftrightharpoons y^2$, $y^2 \in \mathrm{Z}(\Delta^+/O_{19})$ and $\Upsilon \cong \langle x, y^2, z^2\rangle \cong \langle x, z^2\rangle \times \langle y^2\rangle \cong D_3 \times C_2 \cong D_6$.
  In case 20, $(x^2y)^2 = 1$ implies that $(z^2)^x = z^{-2}$, so $\Upsilon(\mathcal{P}_{19}) \cong \langle x, z^2\rangle \cong C_3 \rtimes C_4$.
  In case 21, $(z^2y)^2 = 1$ implies that $z^2 \leftrightharpoons y$. Since $z^2 \leftrightharpoons z$, $z^2 \in \mathrm{Z}(\Delta^+/O_{21})$ and $\Upsilon(\mathcal{P}_{21}) \cong \langle x, z^2\rangle \cong C_6 \times C_2$.
- If j is 1, 2, 3, 4, 5, 6, 8, 9, 12, 13 or 14, $\Upsilon(\mathcal{P}_j) \cong \langle x\rangle \cong C_l$ and ${\mathcal{P}_j}^{\Delta} \cong \mathcal{S}_p$, where $p = |\Omega|/2l$.
- $\Upsilon(\mathcal{P}_{16}) \cong \langle x, z^2\rangle \cong D_4$ because $y^2 = z^4$ implies that $z^x = z^{-5}$ and $(z^2)^x = z^{-10} = z^{-2}$. Since $\mathcal{O}_{16} = \mathcal{H}_{10} \to \mathcal{S}_4$, $\mathcal{P}_{16} = \mathrm{Pin}(\mathcal{O}_{16}) \to \mathrm{Pin}(\mathcal{S}_4) \cong \mathcal{S}_8$ and ${\mathcal{P}_{16}}^{\Delta} = \mathcal{S}_8$.
- $\Upsilon(\mathcal{P}_{23}) \cong \langle x, x^z\rangle \cong Q_8$ since $z \leftrightharpoons x^2$, $(xz)^3 = 1 = (zx)^3$, $(x^z)^2 = (x^2)^z = x^2$ and $(xx^z)^2 = xz(zx)^3 x^{-1} zxz = xzx^{-1}zxz = xzxx^2zxz = x^2(xz)^3 = x^2$. Since $\mathcal{O}_{23} \to \mathcal{S}_3$, $\mathcal{P}_{23} = \mathrm{Pin}(\mathcal{O}_{23}) \to \mathrm{Pin}(\mathcal{S}_3) \cong \mathcal{S}_6$ and ${\mathcal{P}_{23}}^{\Delta} = \mathcal{S}_6$.

Table 5.9 lists the chirality groups, chirality indices and closure covers of the bipartite-regular hypermaps on the double torus obtained by the Pin construction.

5.3.2 Chirality groups and chirality indices of the $\Delta^{+0\hat{0}}$-regular hypermaps on the double torus which are not obtained by the Walsh or Pin constructions

Let $\mathcal{B}_j$ be the $\Delta^{+0\hat{0}}$-regular hypermap listed in line j of Table 5.7 and B_j a hypermap subgroup of $\mathcal{B}_j$. Then $\mathcal{B}_j$ covers $\mathcal{S}_2$ and $\Upsilon(\mathcal{B}_j) = {B_j}^{\Delta}/B_j \lhd \Delta^{+0\hat{0}}/B_j$. As before, let $r = B_j R_1 R_2$, $s = B_j (R_1 R_2)^{R_0}$, $t = B_j (R_2 R_0)^2$ and $u = B_j (R_0 R_1)^2$.

- If j is 1, 3, 5, 6, 11, 13 or 18, then $\mathcal{B}_j$ is regular (see Remark 5.2.3), so $\Upsilon(\mathcal{B}_j) = 1$ and ${\mathcal{B}_j}^{\Delta} = \mathcal{B}_j$.

#	$\mathcal{B} = \mathrm{Pin}(\cdot)$	Υ	ι	$\mathcal{B}^{\Delta}$
1	$\mathrm{Pin}(\mathcal{H}_1)$	C_5	5	$\mathcal{S}_2$
2	$\mathrm{Pin}(\mathcal{H}_2)$	C_5	5	$\mathcal{S}_2$
3	$\mathrm{Pin}(\mathcal{H}_3)$	C_5	5	$\mathcal{S}_2$
4	$\mathrm{Pin}(\mathcal{H}_4)$	C_3	3	$\mathcal{S}_4$
5	$\mathrm{Pin}(\mathrm{D}_{(01)}(\mathcal{H}_4))$	C_6	6	$\mathcal{S}_2$
6	$\mathrm{Pin}(\mathcal{H}_5)$	C_2	2	$\mathcal{S}_8$
7	$\mathrm{Pin}(\mathrm{D}_{(01)}(\mathcal{H}_5))$	C_8	8	$\mathcal{S}_2$
8	$\mathrm{Pin}(\mathcal{H}_6)$	C_4	4	$\mathcal{S}_4$
9	$\mathrm{Pin}(\mathcal{H}_7)$	C_2	2	$\mathcal{S}_{10}$
10	$\mathrm{Pin}(\mathrm{D}_{(01)}(\mathcal{H}_7))$	C_5	5	$\mathcal{S}_4$
11	$\mathrm{Pin}(\mathrm{D}_{(012)}(\mathcal{H}_7))$	C_{10}	10	$\mathcal{S}_2$
12	$\mathrm{Pin}(\mathcal{H}_8)$	C_2	2	$\mathcal{S}_{12}$
13	$\mathrm{Pin}(\mathrm{D}_{(01)}(\mathcal{H}_8))$	C_6	6	$\mathcal{S}_4$
14	$\mathrm{Pin}(\mathcal{H}_9)$	C_3	3	$\mathcal{S}_8$
15	$\mathrm{Pin}(\mathrm{D}_{(01)}(\mathcal{H}_9))$	$C_3 \rtimes C_4$	12	$\mathcal{S}_2$
16	$\mathrm{Pin}(\mathcal{H}_{10})$	D_4	8	$\mathcal{S}_4$
17	$\mathrm{Pin}(\mathrm{D}_{(01)}(\mathcal{H}_{10}))$	Q_8	8	$\mathcal{S}_4$
18	$\mathrm{Pin}(\mathrm{D}_{(012)}(\mathcal{H}_{10}))$	C_8	8	$\mathcal{S}_4$
19	$\mathrm{Pin}(\mathcal{H}_{11})$	D_6	12	$\mathcal{S}_4$
20	$\mathrm{Pin}(\mathrm{D}_{(01)}(\mathcal{H}_{11}))$	$C_3 \rtimes C_4$	12	$\mathcal{S}_4$
21	$\mathrm{Pin}(\mathrm{D}_{(012)}(\mathcal{H}_{11}))$	$C_6 \times C_2$	12	$\mathcal{S}_4$
22	$\mathrm{Pin}(\mathcal{H}_{12})$	$SL_2(3)$	24	$\mathcal{S}_2$
23	$\mathrm{Pin}(\mathrm{D}_{(02)}(\mathcal{H}_{12}))$	Q_8	8	$\mathcal{S}_6$
24	$\mathrm{Pin}(\mathcal{H}_{13})$	$GL_2(3)$	48	$\mathcal{S}_2$
25	$\mathrm{Pin}(\mathrm{D}_{(01)}(\mathcal{H}_{13}))$	$SL_2(3)$	24	$\mathcal{S}_4$
26	$\mathrm{Pin}(\mathrm{D}_{(012)}(\mathcal{H}_{13}))$	$GL_2(3)$	48	$\mathcal{S}_2$

Table 5.9: Bipartite-regular hypermaps on the double torus obtained by the Pin construction.

#	$\|\Omega\|$	$\mathcal{H}$	bip.-type	ι	Υ	$\mathcal{H}^\Delta$
1	12	$\mathcal{B}_1$	$(3,3;6,6)$	1	1	$\mathcal{H}_4$
2	12	$\mathcal{B}_2$	$(3,3;6;6)$	3	C_3	$\mathcal{S}_2$
3	16	$\mathcal{B}_3$	$(2,2;8;8)$	1	1	$\mathcal{H}_5$
4	16	$\mathcal{B}_4$	$(2,4;4;8)$	4	C_4	$\mathcal{S}_2$
5	16	$\mathcal{B}_5$	$(4,4;4;4)$	1	1	$\mathcal{H}_6$
6	24	$\mathcal{B}_6$	$(2,2;6;6)$	1	1	$\mathcal{H}_8$
7	24	$\mathcal{B}_7$	$(2,2;6;6)$	3	C_3	$\mathcal{P}_2$
8	24	$\mathcal{B}_8$	$(2,3;4;6)$	6	C_6	$\mathcal{S}_2$
9	24	$\mathcal{B}_9$	$(2,3;4;6)$	6	D_3	$\mathcal{S}_2$
10	24	$\mathcal{B}_{10}$	$(2,3;4;6)$	6	D_3	$\mathcal{S}_2$
11	24	$\mathcal{B}_{11}$	$(3,3;4;4)$	1	1	$\mathcal{H}_9$
12	24	$\mathcal{B}_{12}$	$(3,3;4;4)$	3	C_3	$\mathcal{S}_4$
13	32	$\mathcal{B}_{13}$	$(2,2;4;8)$	1	1	$\mathcal{H}_{10}$
14	32	$\mathcal{B}_{14}$	$(2,2;4;8)$	4	V_4	$\mathcal{S}_4$
15	32	$\mathcal{B}_{15}$	$(2,4;4;4)$	2	C_2	$\mathrm{D}_{(12)}(\mathcal{P}_4)$
16	32	$\mathcal{B}_{16}$	$(2,4;4;4)$	2	C_2	$\mathcal{P}_4$
17	32	$\mathcal{B}_{17}$	$(2,4;4;4)$	4	C_4	$\mathcal{S}_4$
18	48	$\mathcal{B}_{18}$	$(2,2;4;6)$	1	1	$\mathcal{H}_{11}$
19	48	$\mathcal{B}_{19}$	$(2,2;4;6)$	3	C_3	$\mathrm{D}_{(12)}(\mathcal{P}_4)$
20	48	$\mathcal{B}_{20}$	$(2,3;4;4)$	6	D_3	$\mathcal{S}_4$
21	48	$\mathcal{B}_{21}$	$(2,3;4;4)$	6	D_3	$\mathcal{S}_4$
22	48	$\mathcal{B}_{22}$	$(2,3;4;4)$	6	C_6	$\mathcal{S}_4$

Table 5.10: $\Delta^{+0\hat{0}}$-regular hypermaps on the double torus which are not obtained by the Pin or the Walsh constructions

- If j is 2, 4, 8, 9 or 10, $\Upsilon(\mathcal{B}_j) = \Delta^{+0\hat{0}}/B_j$, because $\Upsilon(\mathcal{B}_j)$ and $\Delta^{+0\hat{0}}/B_j$ have the same order. In addition, ${\mathcal{B}_j}^\Delta = \mathcal{S}_2$.
- If j is 7, 12, 15, 16 or 19, $\Upsilon(\mathcal{B}_j)$ has prime order and hence is cyclic. An easy calculation reveals that $\mathcal{B}_7$ covers $\mathcal{P}_2$, $\mathcal{B}_{12}$ covers $\mathcal{S}_4$, and $\mathcal{B}_{15}$ and $\mathcal{B}_{19}$ cover $\mathrm{D}_{(12)}(\mathcal{P}_2)$. Owing to this, ${\mathcal{B}_7}^\Delta = \mathcal{P}_2$, ${\mathcal{B}_{12}}^\Delta = \mathcal{S}_4$ and ${\mathcal{B}_{15}}^\Delta = {\mathcal{B}_{19}}^\Delta = \mathrm{D}_{(12)}(\mathcal{P}_4)$. The closure cover of $\mathcal{B}_{16} \cong \mathrm{D}_{(12)}(\mathcal{B}_{15})$ is ${\mathcal{B}_{16}}^\Delta \cong \mathrm{D}_{(12)}(\mathcal{B}_{15})^\Delta \cong \mathrm{D}_{(12)}({\mathcal{B}_{15}}^\Delta) \cong \mathrm{D}_{(12)}(\mathrm{D}_{(12)}(\mathcal{P}_4)) \cong \mathcal{P}_4$.
- $\Upsilon(\mathcal{B}_{14}) = \langle r, u^2 \rangle \cong V_4$ and ${\mathcal{B}_{14}}^\Delta = \mathcal{P}_2$ because ${\mathcal{B}_{14}}^\Delta$ has 8 flags and covers $\mathcal{P}_2$.
- $\Upsilon(\mathcal{B}_{17}) = \langle s \rangle \cong C_4$ and ${\mathcal{B}_{17}}^\Delta = \mathcal{S}_4$ because ${\mathcal{B}_{17}}^\Delta$ has 8 flags and covers $\mathcal{S}_4$.
- If j is 20, 21 or 22, $\Upsilon(\mathcal{B}_j) = \langle r, s \rangle$ and ${\mathcal{B}_j}^\Delta = \mathcal{S}_4$. When j is 22, $r \leftrightharpoons s$ and so $\Upsilon(\mathcal{B}_{22}) \cong C_2 \times C_3 \cong C_6$. When j is 20 or 21, $rs \neq sr$ and hence $\Upsilon(\mathcal{B}_j) \cong D_3$.

Table 5.10 displays the chirality groups, chirality indices and closure covers of the 2-restrictedly-regular hypermaps on the double torus which are not obtained by the Walsh or Pin constructions.

Appendix A

Normal closures, cores and homomorphisms

We list here some results about group theory used in the thesis.

In what follows we assume that G and G' are groups. As mentioned before, the normalizer of H in G is be denoted by $\mathrm{N}_G(H)$ and the center of G is denoted by $\mathrm{Z}(G)$. The kernel of a group homomorphism $\varphi : G \to G'$ is denoted by $\ker \varphi$.

Proposition A.1.1. *Let $\varphi : G \to G'$ be a group homomorphism and H' a subgroup of G'. Then:*

1. $[G : H'\varphi^{-1}] \leq [G' : H']$.
2. *If φ is onto, then $[G : H'\varphi^{-1}] = [G' : H']$.*
3. *If φ is onto and $H' \lhd G'$, $G/H'\varphi^{-1}$ is isomorphic to G'/H'.*

Applying Proposition A.1.1 to the inclusion $\iota : H \to G$ and to the projection $\pi : G \to G/N$, we get the following result.

Corollary A.1.2. *Let H and N be subgroups of G. Then:*

1. $[H : H \cap N] \leq [G : N]$.
2. *If N is normal in G, then $[H : H \cap N] = [G : N]$ if and only if $G = HN$.*
3. *If N is normal in G and $N \subseteq H$, then $[G : H] = [G/N : H/N]$.*
4. *If $[G : N] = 2$ and $H \nsubseteq N$, then $[H : H \cap N] = 2$.*

The following result comes as Exercise 9 in page 75 of [50] and as Exercise 1.1.2 in page 3 of [60].

Lemma A.1.3.

1. *Let H be a subgroup of G of finite index. Then there is a normal subgroup N of G contained in H and also of finite index.*
2. *Let H and H' be subgroups of G of finite index. Then $H \cap H'$ also has finite index.*

As a by-product of the proof of Lemma A.1.3, we get:

Remark A.1.4. If H is a subgroup of G of finite index, then H^G has finite index, because $[G : H^G] \leq [G : H^G] \cdot [H^G : H] = [G : H]$, and H_G has finite index by the previous lemma.

It is easy to see that H_G is a normal subgroup of H and that H is a normal subgroup of $\mathrm{N}_G(H)$. However H may not be normal in H^G. The following result gives us a necessary and sufficient condition for a subgroup H to be normal in its closure cover H^G.

Lemma A.1.5. *Let H be a subgroup of G. Then H is normal in H^G if and only if there is a normal subgroup N of G such that H is normal in N.*

Proposition A.1.6. *Let G be a group, N a normal subgroup of G and H a subgroup of G such that $H \subseteq N$.*

1. $H^N \subseteq H^G$ *and* $H_N \supseteq H_G$.
2. $\left(H^N\right)^G = H^G$ *and* $\left(H_N\right)_G = H_G$.
3. *For all* $g \in G$, $(H^g)^N = \left(H^N\right)^g$.
4. $\mathrm{N}_G(H) \subseteq \mathrm{N}_G(H^N)$.

Lemma A.1.7. *Let N be a normal subgroup of G of index 2, $k \in G \setminus N$ and H a normal subgroup of N. Then $H_G = H \cap H^k$ and $H^G = HH^k$.*

Proposition A.1.8. *Let $\varphi : G \to G'$ be a group homomorphism, $H \leq G$ and $H' \leq G'$. Then:*

1. $\left(H'\varphi^{-1}\right)^G \subseteq \left(H'^{G'}\right)\varphi^{-1}$ *and* $\left(H'\varphi^{-1}\right)_G \supseteq \left(H'_{G'}\right)\varphi^{-1}$.
2. *If φ is an epimorphism, then* $\left(H'\varphi^{-1}\right)^G \supseteq \left(H'^{G'}\right)\varphi^{-1}$ *and* $\left(H'\varphi^{-1}\right)_G \subseteq \left(H'_{G'}\right)\varphi^{-1}$.
3. $(H\varphi)^{G'} \supseteq \left(H^G\right)\varphi$; *if* $H \supseteq \ker\varphi$, *then* $(H\varphi)_{G'} \subseteq \left(H_G\right)\varphi$.
4. *If φ is an epimorphism, then* $(H\varphi)^{G'} \subseteq \left(H^G\right)\varphi$ *and* $(H\varphi)_{G'} \supseteq \left(H_G\right)\varphi$.

Corollary A.1.9. *Let $\varphi : G \to G'$ be an epimorphism, $H \leq G$ and $H' \leq G'$. Then:*

1. $(H'\varphi^{-1})^G = (H'^{G'})\varphi^{-1}$ *and* $(H'\varphi^{-1})_G = (H'_{G'})\varphi^{-1}$.
2. $(H\varphi)^{G'} = (H^G)\varphi$; *if φ is an isomorphism, then* $(H\varphi)_{G'} = (H_G)\varphi$.

When φ is an inner automorphism of G we get:

Corollary A.1.10. *For all $g \in G$, $H_G = (H^g)_G$ and $H^G = (H^g)^G$.*

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